# Einführung in die Quantenoptik II

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**Übungsaufgaben Blatt 1** Ausgabe: 01. April 2019
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**Aufgabe 1.1** – Positive maps and the Kraus representation (10 Punkte)

In the lecture, we have encountered dynamical (or completely positive = CP) maps. Recall that such a map $\Lambda : \rho \mapsto \Lambda(\rho)$

- is linear: $\Lambda(p\rho + q\rho') = p\Lambda(\rho) + q\Lambda(\rho')$ for real $p, q \geq 0$
- is positive and preserves the trace: $\langle\psi|\Lambda(\rho)|\psi\rangle \geq 0$ for all $|\psi\rangle$ and all $\rho$, $\mathrm{tr}\,\Lambda(\rho) = \mathrm{tr}\,\rho$.
- is completely positive: if $\Lambda$ is extended to $\Lambda \otimes \mathbb{1}$ on a larger Hilbert space, then this map is positive.

(1) Show that unitary time evolution for a closed system, $\rho \mapsto U\rho U^\dagger$ with unitary $U$, is a CP map.
(2) Show that CP maps form a convex set: $p\Lambda + q\Lambda'$ is CP for real $0 \leq p, q \leq 1$, $p + q = 1$ if $\Lambda$ and $\Lambda'$ are CP.
(3) Conclude that the 'Kraus operation' defined as follows is a CP map:

$$\rho \mapsto \sum_k p_k U_k \rho U_k^\dagger\,, \quad \sum_k p_k = 1\,, \quad U_k \text{ unitary} \tag{1.1}$$

**Aufgabe 1.2** – Positive maps and transposition (10 Punkte)

Transposition is an example of a positive, but not completely positive map. To see this, consider a two-level system and define the transposed density matrix in a given basis $|a\rangle, a = e, g$, by

$$\rho = \sum_{a,b} \rho_{ab} |a\rangle\langle b| \mapsto \rho^{\mathrm{T}} = \sum_{a,b} \rho_{ab} |b\rangle\langle a| \tag{1.2}$$

(1) Show that the matrix representing $\rho^{\mathrm{T}}$ is the transpose of $\rho$; its matrix elements are the complex conjugates of those of $\rho$: $(\rho^{\mathrm{T}})_{ab} = (\rho_{ab})^*$. Show that $\rho^{\mathrm{T}}$ is again a density operator.

(2) For a system of two qubits, take the basis $\{|ee\rangle, |ge\rangle, |eg\rangle, |gg\rangle\}$ and consider in this basis a two-qubit density matrix (greek capital $\rho$)

$$P = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{1.3}$$

with $2 \times 2$ block matrices $A, B, C, D$. Check that $A^\dagger = A$, $D^\dagger = D$, $B^\dagger = C$, $\mathrm{tr}(A + D) = 1$. Calculate the reduced density matrices $\mathrm{tr}_2(P)$ and $\mathrm{tr}_1(P)$ for the first and the second qubit (the index at the trace means: 'trace out this sub-system')

$$\mathrm{tr}_2(P) = A + D\,, \qquad \mathrm{tr}_1(P) = \begin{pmatrix} \mathrm{tr}\, A & \mathrm{tr}\, B \\ \mathrm{tr}\, C & \mathrm{tr}\, D \end{pmatrix} \tag{1.4}$$

(3) The transposed density matrix is of course

$$P^{\mathrm{T}} = \begin{pmatrix} A^{\mathrm{T}} & C^{\mathrm{T}} \\ B^{\mathrm{T}} & D^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} A^* & B^* \\ C^* & D^* \end{pmatrix} \tag{1.5}$$

The *partial transpose* $P^\Gamma$ ('a half of a $T$', applied to the *second* system) is the operator defined by

$$P = \sum_{a,b,c,d} P_{ac,bd} |a\, c\rangle\langle b\, d| \mapsto P^\Gamma = \sum_{a,b,c,d} P_{ac,bd} |a\, d\rangle\langle b\, c| \tag{1.6}$$

Show that for two qubits, its matrix representation is

$$P^\Gamma = \begin{pmatrix} A & C \\ B & D \end{pmatrix} \tag{1.7}$$

'This is intuitive': the reduced state for the second qubit, $\mathrm{tr}_1(P^\Gamma)$, is the transpose of $\mathrm{tr}_1(P)$, while nothing changes when we trace out the second qubit.

The partial transpose on the first system, which we denote $P^{⅂}$, can be computed by the 'total transpose' of $P^\Gamma$

$$P^{⅂} = (P^\Gamma)^{\mathrm{T}} = \begin{pmatrix} A^{\mathrm{T}} & B^{\mathrm{T}} \\ C^{\mathrm{T}} & D^{\mathrm{T}} \end{pmatrix} = \begin{pmatrix} A^* & C^* \\ B^* & D^* \end{pmatrix} \tag{1.8}$$

(4) Show that all these maps are involutions: apply them twice, and nothing has changed.

(5) Consider the state (this one is called 'completely entangled', a projector on $|ge\rangle + |eg\rangle$)

$$P = \frac{1}{2} \begin{pmatrix} 0 & & & \\ & 1 & 1 & \\ & 1 & 1 & \\ & & & 0 \end{pmatrix} \tag{1.9}$$

where non-written matrix elements are zero, and show that its partial transposes $P^\Gamma = P^\text{T}$ are *not* positive.

**Aufgabe 1.3** – Lindblad master equation for operator averages (5 Punkte)
In the lecture, we have seen the Lindblad form of the master equation

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[H, \rho] + \sum_k \left( L_k \rho L_k^\dagger - \tfrac{1}{2}\{\rho L_k^\dagger L_k + L_k^\dagger L_k \rho\} \right) \tag{1.10}$$

where $H$ is the hermitean Hamilton operator and $L$ are the Lindblad operators.

(i) This is called the diagonal form, but there are more general forms. Play with the *Ansatz*

$$\sum_{kl} \gamma_{kl} \left( A_k \rho A_l^\dagger - \tfrac{1}{2}\{\rho A_l^\dagger A_k + A_l^\dagger A_k \rho\} \right) \tag{1.11}$$

for the Lindblad terms where the $A_k$ are any system operators. Show that $\rho$ remains hermitean under this master equation provided that $(\gamma_{kl})$ is a hermitean matrix. Diagonalize this matrix and show that one gets back the diagonal Lindblad form (1.10) provided that the eigenvalues of $(\gamma_{kl})$ are non-negative. (In that case, the matrix $(\gamma_{kl})$ itself is called non-negative or, slightly less precise, positive.)

(ii) In the following snippet of python code,

```
def decay_deph_e(rho, gamma_e = gmma_1, gamma_d = Gmma_2 - 0.5*gmma_1):
    if gamma_d < 0:
        print('!!Warning!! Dephasing rate Gmma_2 too small. Override applied.')
        gamma_d = Gmma_2
    # spontaneous decay and dephasing of emitter
    return gamma_e * S.dot(rho).dot(Sdag) \
           + 2*gamma_d * nbS.dot(rho).dot(nbS) \
           - (gamma_d + 0.5*gamma_e)*(nbS.dot(rho) + rho.dot(nbS))
```

certain terms in the master equation are defined. Here, $S = |g\rangle\langle e|$ is the lowering operator for a two-level system, `nbS` $= S^\dagger S$ and `.dot` is the matrix product. Bring these terms into the Lindblad form using two Lindblad operators $L_1$ and $L_2$. Justify the warning in the code.

(iii) One is often interested in the time evolution of certain observables, say $A$, of the system. Show from the master equation (1.10) that its average evolves according to

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle A\rangle = \frac{\mathrm{i}}{\hbar}\langle [H, A]\rangle + \sum_k \tfrac{1}{2}\big\langle \big[L_k^\dagger, A\big] L_k \big\rangle + \tfrac{1}{2}\big\langle L_k^\dagger \big[A, L_k\big]\big\rangle \tag{1.12}$$

a generalisation of the Heisenberg picture. Work out the equation of motion for $S$ and $S^\dagger$ from question (ii). You may take $L_1 \propto S$.

**Aufgabe 1.4** – Journal templates in physics (8 Punkte)

Scientific journals propose to their authors "templates" for writing papers. Check out the journals listed below and find their templates on the web.

Each student takes a different template and completes it with the following information: Title, author, affiliation, abstract and a bibliography list with three items. If you need inspiration, just produce a fake copy of the paper by Einstein, Podolski and Rosen (1935) about the incompleteness of quantum mechanics. Print out the result and hand it in with the rest of your problem solutions. For one of the problems in this semester, you will be asked to hand in a solution in a similar electronic form (this time under your name).

Nature Photonics, Europhysics Letters, Optics Letters, Physical Review A, Journal of Physics B, European Physical Journal D, Journal of Optics A, Journal of modern Optics, Optics Communications, Journal of the Optical Society of America B, Annalen der Physik (Berlin), Annals of Physics (N.Y.)

[Bonus points if your bibliography is "correctly" formatted.]