Einführung in die Quantenoptik II

Sommersemester 2020

Carsten Henkel

Übungsaufgaben Blatt 4

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Aufgabe 4.1 – Spectrum of Resonance Fluorescence (10 Punkte)

We shall see in the lecture that the field emitted by a two-level atom can be described by the positive-frequency field operator

$$E(\mathbf{r}, t) \sim \frac{\mathrm{e}^{-\mathrm{i}\omega_L t} d_{ge} \sigma(t)}{r} \tag{4.1}$$

where $\sigma(t) = \frac{1}{2}(\sigma_1(t) - \mathrm{i}\sigma_2(t))$ is the lowering operator of the two-level system. (In the Schrödinger picture, $\sigma = |g\rangle\langle e|$.)

The light emitted by the atom is called *fluorescence* (or luminescence) and its spectrum is proportional to the Fourier transform of the atomic correlation function $C_\sigma(\tau) = \langle \sigma^\dagger(t+\tau)\sigma(t)\rangle$ with respect to τ. (We assume that t is large enough so that the system is in a steady state.) When this correlation is computed with the quantum regression formula, we introduce the *skew* density operator $\varrho(\tau|\sigma\rho_{\mathrm{st}})$ that evolves according to the master equation and from the initial condition $\varrho(0|\sigma\rho_{\mathrm{st}}) = \sigma\rho_{\mathrm{st}}$.

(1) Show that $\varrho(0)$ has the trace $(s_1 - \mathrm{i}s_2)/2$ and is described by the complex Bloch vector $\mathbf{S}$ with components (why is $\mathbf{S}$ complex?)

$$S_1 = \frac{1+s_3}{2}, \qquad S_2 = \frac{1+s_3}{2\mathrm{i}}, \qquad S_3 = \frac{-s_1 + \mathrm{i}s_2}{2} \tag{4.2}$$

Here, $\mathbf{s} = (s_1, s_2, s_3)$ is the steady-state Bloch vector corresponding to ρ_{st}.

(2) The time evolution of the Bloch vector is given by the Bloch equations

$$\frac{\mathrm{d}S_1}{\mathrm{d}\tau} = -\Gamma S_1 + \Delta S_2 \qquad \frac{\mathrm{d}S_2}{\mathrm{d}\tau} = -\Delta S_1 - \Gamma S_2 - \Omega S_3$$
$$\frac{\mathrm{d}S_3}{\mathrm{d}\tau} = \Omega S_2 - \gamma(S_3 + \mathrm{tr}\,\varrho) \tag{4.3}$$

where Δ is the detuning, Ω the (real) Rabi frequency and Γ and γ the decay rates for coherences and populations. Argue that the stationary solution of these equations is given by $\varrho_{\mathrm{st}} = (\mathrm{tr}\,\varrho)\rho_{\mathrm{st}}$. Check that in the stationary state P_{st}, the expectation value of $\sigma^\dagger = (\sigma_1 + \mathrm{i}\sigma_2)/2$ is given by

$$\mathrm{tr}(\sigma^\dagger P_{\mathrm{st}}) = |\mathrm{tr}(\sigma\rho_{\mathrm{st}})|^2 \tag{4.4}$$

What does this stationary value tell you about the behaviour of the correlation function $C_\sigma(\tau)$ as $\tau \to \infty$? What does this imply for the fluorescence spectrum?

Answer: there is an "elastic" component in the spectrum that is monochromatic and centered at the driving laser frequency ω_L. Its power is proportional to Eq.(4.4).

(3) Eqs.(4.3) can be written as $\dot{\mathrm{S}} = -\mathbf{B}\mathrm{S} + \ldots$ with a 3×3 matrix **B**. Show that the characteristic equation for its eigenvalues λ can be written as

$$(\Gamma - \lambda)^2(\gamma - \lambda) + \Omega^2(\Gamma - \lambda) + \Delta^2(\gamma - \lambda) = 0 \tag{4.5}$$

Solve this equation approximately in the two cases $|\Omega| \ll \gamma/2 < \Gamma$ and $|\Omega| \gg \Gamma$. It turns out that the real and imaginary parts of the eigenvalues λ describe the width and the position of spectral lines – what kind of behaviour for the fluorescence spectrum do you expect in the two cases?

Aufgabe 4.2 – Anti-bunching (10 Punkte)

The correlations of the fluorescence intensity I is related to the probability of detecting two photons, one at t and another one at the later time $t' = t + \tau$. According to Glauber's photodetector theory, the joint detection rate is proportional to the correlation function (time and normal order)

$$C_I(\tau) = \langle \sigma^\dagger(t)\sigma^\dagger(t+\tau)\sigma(t+\tau)\sigma(t)\rangle \tag{4.6}$$

where we have used Eq.(4.1) for the emitted field.

(1) Construct with the regression approach a formula that gives this 'intensity correlation' in terms of a conditional density operator $P(\tau|\ldots)$ (different from the one used in Problem 4.1). Justify the wording "just after the time t, the two-level system is for sure in its ground state" and check that the initial state for $P(\tau|\ldots)$ is (nearly) an ordinary density operator.

(2) Show that for $\tau = 0$, $C_I(0) = 0$ and give an interpretation for this result.

(3) In a classical description, the fluorescence intensity $I(t)$ is proportional to $|d(t)|^2$ where d is a complex dipole amplitude. Show with the Schwarz inequality that in this picture, the correlation function satisfies

$$\langle |d(t)|^4\rangle \geq \langle |d(t)|^2|d(t')|^2\rangle \quad \text{and} \quad \langle |d(t)|^4\rangle \geq \langle |d(t)|^2\rangle^2 \tag{4.7}$$

where the averages are taken with respect to a probability distribution for $d(t')$. We assume that averages are stationary.

Hint. For real random variables, A, B, we have $0 \leq \langle (A-B)^2\rangle = \langle A^2\rangle + \langle B^2\rangle - 2\langle AB\rangle$ and $0 \leq \langle (A - \langle A\rangle)^2\rangle$.

For this reason, the quantum result $C_I(0) = 0$ is called "non-classical": it cannot be reproduced by replacing the quantum dipole operator σ by classical variables with whatever distribution function.