

Chapter 2

Quantum states of light

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2.1 Wigner & co: phase space distributions

The states of a quantum system can not only be characterized by their wave functions (or density operators), but also by their behaviour in phase space. Fig. 2.1 is one example for a coherent state, other states known in quantum optics are illustrated in Fig. 2.3.

There are several possibilities to construct distribution functions on the 'phase space' spanned by the quadratures X and P . This is rooted in the fact that these are non-commuting operators.

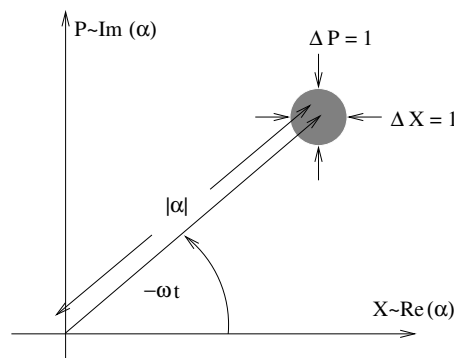


Figure 2.1: Q-function of a coherent state in phase space.

2.1.1 The Wigner function

The traditional definition in phase space is

$$W(x, p) = \int \frac{dy}{2\pi\hbar} e^{-ipy/\hbar} \psi(x + y/2) \psi^*(x - y/2) \quad (2.1)$$

where $\psi(x)$ is the wave function of a system.

Remember that phase space is spanned by position x and momentum p , a pair of canonically conjugate coordinates. In quantum optics, the electric and magnetic field amplitudes of one mode provide another example. Two quadratures, shifted in phase by 90° , are another choice. In the following, we use the language of ordinary quantum mechanics and stick to a one-dimensional ‘configuration space’ (the number of dimensions of the x -variable).

The definition (2.1) tries to capture information from both variables: with respect to x , the position distribution. The integral over y tries to include aspects of the momentum distribution.

Wigner function is real:

$$\begin{aligned} W^*(x, p) &= \int \frac{dy}{2\pi\hbar} e^{ipy/\hbar} \psi^*(x + y/2) \psi(x - y/2) \\ &= \int \frac{dy}{2\pi\hbar} e^{-ipy/\hbar} \psi^*(x - y/2) \psi(x + y/2) \quad (\text{substitute } y \mapsto -y) \\ &= W(x, p) \end{aligned} \quad (2.2)$$

Simple exercises.

1. A Gaussian wavepacket, $\psi(x) = N \exp(ikx - x^2/4\sigma^2)$ gives a Wigner function which is a double Gaussian,

$$W(x, p) = N' \exp(-x^2/2\sigma^2 - a(p - \hbar k)^2) \quad (2.3)$$

with constants N' and a to be calculated. Are there nicer choices for the exponent $x^2/4\sigma^2$ in the Gaussian? Observe that this is a “minimum uncertainty” wave packet, i.e., $\Delta x \Delta p \sim \hbar$.

2. A wave function that is odd in x , as it happens for stationary states in symmetric potentials, gives a Wigner function that is strictly negative for $x = 0 = p$.

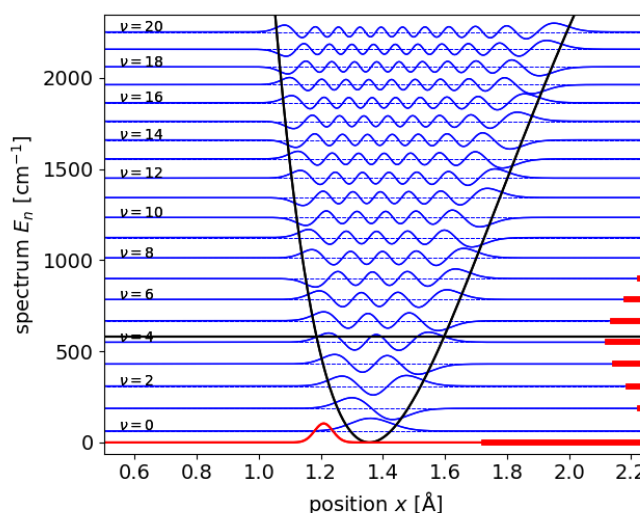
3. The integral of the Wigner function over all momenta p gives the position distribution

$$|\psi(x)|^2 = \int dp W(x, p) \quad (2.4)$$

This can be visualised as the “shadow” of the Wigner function, represented in three dimensions as a “landscape” on the phase space plane, the shadow being cast on the x -axis. This picture is not completely correct because the integral is a sum, and we shall see that positive and negative values of W may combine to zero here.

Examples from molecular physics.

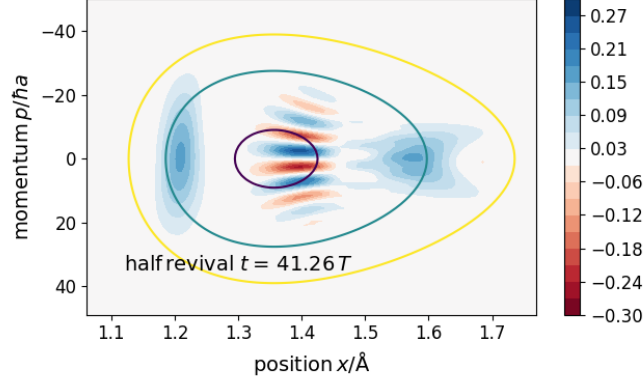
The following picture illustrates the vibrational states of a bi-atomic molecule.



Note the asymmetric shape of the potential (black line): this is typical for a covalent bond – as the nuclei move apart, the electronic wave functions overlap less, the binding force is getting smaller. As they move closer together, Coulomb and Pauli repulsion make the bond ‘stiffer’.

These states live on an “excited potential surface” where the electronic wave function differs from the ground state. The red curve illustrates the ground state in the ground-state potential. With a short laser pulse (duration of a few fs only), one can excite the molecule and “lift the wave packet” of the nuclei onto the excited potential. There, it can be expanded over some eigenstates (the corresponding amplitudes are illustrated by the

red bars on the side). The wave packet is thus not a stationary state, and it will oscillate back and forth in the potential. Its average energy is given by the black line.



Wigner function of the wave packet after a few oscillations. The oval lines are contours of constant energy. A classical particle in this potential would move along one of these lines. The energy of the blue line corresponds to the mean energy of the wave packet (black line above). One sees that the wave packet is split in two or three components: two of them (blue shade, positive values) are located at the “turning points” (the momentum p) is around zero. The colored structure in the middle is called the “ghost component”. One can show (see exercise below) that whenever one tries to measure the particle in this region of phase space (this can be done by computing the overlap with a Gaussian wave packet of Exercise 1. above), the probability of finding the particle there is zero. The Indian physicist Sudarshan has used the word “Tamas component” from a Hindu concept of “dark” or “empty”.

If we compute the projection of the Wigner function onto the position axis, it turns out that one finds two maxima at the turning points, and the red-blue shaded area gives zero. This projection is a degenerate form of computing the overlap with a very elongated wave packet (very narrow in x , very wide in p).

The momentum distribution is given by a formula similar to (2.4)

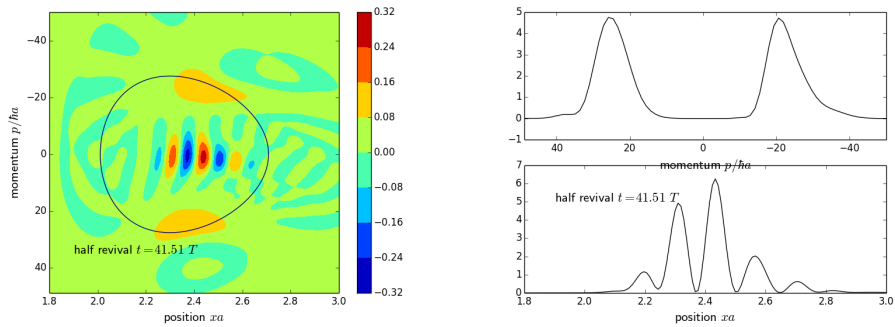
$$|\tilde{\psi}(p)|^2 = N \int dx W(x, p) \quad (2.5)$$

where $\tilde{\psi}(p)$ is the Fourier transform of $\psi(x)$ and the normalisation N depends on the convention for the Fourier transform. One find by projecting

the Wigner function above that the momentum distribution shows oscillations. This can be interpreted as the interference pattern from two slits – indeed, this pattern is the Fourier transform of the wave function. The positive and negative values of the ghost component are thus related to quantum interference, they produce the characteristic bright and dark fringes.

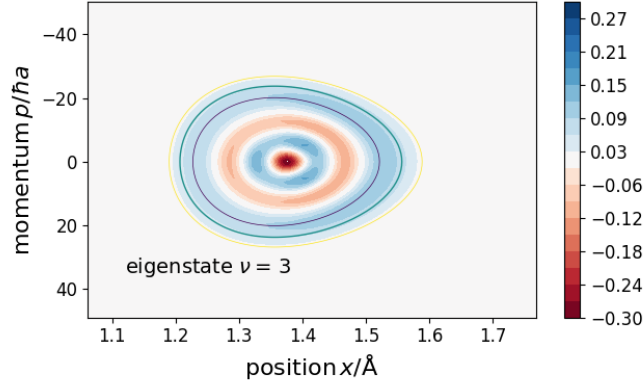
Exercise 4. Choose your favourite convention for the Fourier transform and compute N .

The two pictures here give the Wigner function (with a different color scheme) a quarter of the period T later.



The two maxima of the wave packet have moved to large and small momenta, and the orientation of the colored fringes has changed. Now the position distribution shows fringes, while the momentum distribution has two peaks. In optics, this would correspond to two light beams that from some angle and that are brought to interfere. In their overlap region, the sum of waves going to the right and left produces a standing wave with a modulated intensity in position.

The following picture gives the Wigner function of a stationary state, here an odd eigenstate with quantum number $\nu = 3$ in the excited state potential. (Since the potential is not exactly symmetric, the eigenstate is not exactly odd, although this is difficult to see from the wave functions above.)



See that now the red and blue fringes have a different pattern: they are organised along contours of constant energy. The middle, solid contour that is drawn here corresponds to the energy eigenvalue of this state, the outer and inner contours to the neighboring eigenvalues. In the center, the red spot marks a region of negative values for the Wigner function.

Exercise 5. Show that the squared overlap between two wave functions ψ_1 and ψ_2 is related to an overlap integral between their Wigner functions W_1 and W_2 :

$$|\langle \psi_2 | \psi_1 \rangle|^2 = N \int dx dp W_1(x, p) W_2(x, p) \quad (2.6)$$

and compute the constant N . This formula is the overlap theorem mentioned above: the integral of a Wigner function with a minimum-uncertainty Gaussian is positive. The theorem is, by the way, the mathematical expression of a principle that is often used in statistical physics: if we want to “count” how many quantum states can be put into some region of phase space, then each state occupies an “area of order $2\pi\hbar$ ”.

For the stationary states of a harmonic oscillator, the Wigner function can be computed explicitly. This calculation is nontrivial, and the formula gives no guarantee for consistent prefactors

$$W_n(x, p) = (-1)^n e^{-x^2 - p^2} L_n(2x^2 + 2p^2) \quad (2.7)$$

where $L_n(\cdot)$ is the Laguerre polynomial.¹ This is written in dimensionless coordinates for the harmonic oscillator ($\hbar = m = \omega = 1$). See Fig.2.2 where we have set $|\alpha|^2 = (x^2 + p^2)/2$.

¹Eq.(2.7) with the Laguerre polynomial can be derived from Eq.(2.34) and the gener-

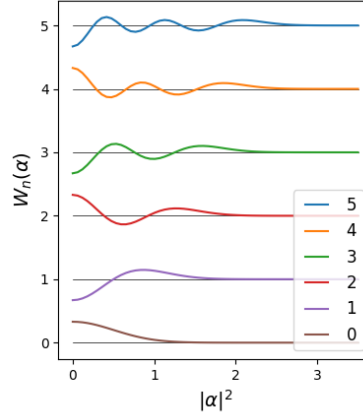


Figure 2.2: Radial cut through the Wigner function of the stationary states of a harmonic oscillator (Fock states or number states in quantum optics).

Exercise 6. The Wigner function is the “Fourier transform of the expectation value of a displacement operator”:

$$W(x, p) = \int \frac{dk ds}{(2\pi)^2 \hbar} \langle \hat{D}(k, s) \rangle \exp[i(kx - ps/\hbar)] \quad (2.8)$$

$$\hat{D}(k, s) = \exp[-i(k\hat{x} - \hat{p}s/\hbar)] \quad (2.9)$$

where \hat{x} , \hat{p} are the position and momentum operators.

(a) To prove this relation, use the Baker-Campbell-Hausdorff relation² for operators A and B

$$e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B \quad (2.10)$$

that both commute with their commutator $[A, B]$. Construct B from the momentum operator \hat{p} and use the fact that an exponential of \hat{p} is a displacement in position space.

(b) In quantum optics, the displacement operator is defined by Glauber according to

$$\hat{D}(\alpha) = \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}) \quad (2.11)$$

ating function [Eq.(18.12.13) of <http://dlmf.nist.gov>]

$$(1-z)^{-1} \exp\left(\frac{xz}{z-1}\right) = \sum_{n=0}^{\infty} L_n(x) z^n,$$

using $z = -e^{-\hbar\omega/k_B T}$ and $\bar{n} + \frac{1}{2} = \frac{1}{2}(1-z)/(1+z)$.

²It is actually unfair to use the names of Baker, Campbell and Hausdorff for this formula which goes back to Glauber anyway. The other three formulated an expression that is valid in the general case that A , B and $[A, B]$ and ... do not commute.

where \hat{a} and \hat{a}^\dagger are the (dimensionless) annihilation and creation operators. Show that with a suitable choice of the complex number α , this is a special case of $\hat{D}(k, s)$ defined in (2.9).

2.1.2 Measuring the Wigner function: tomography

The Wigner function for a point particle can be measured by using a technique called “tomography”. This is familiar from medical imagery where a three-dimensional distribution is reconstructed from two-dimensional “shadow images”. The key point is that one needs projections along different “directions” in phase space, mathematically speaking along arbitrary angles. In the projection theorems, we have seen projections onto the position and momentum axis. Without going into the details of the tomographic reconstruction, it may suffice to point out how one may measure relatively easily projections (or “slices”) of the Wigner function along tilted axes.

The basic idea is to let the particle evolve “freely”, without any external potential. The technique is sometimes called “time of flight” because in an experiment, particles are released to expand freely. Since the kinetic energy is quadratic in momentum, one can show that the time evolution of the Wigner function is

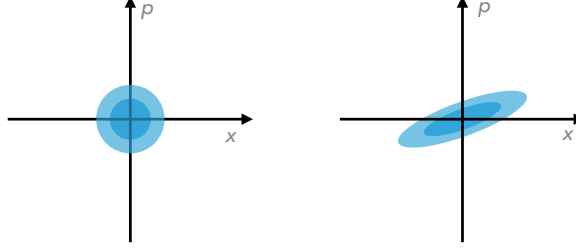
$$\partial_t W + \frac{p}{m} \partial_x W = 0 \quad (2.12)$$

This looks very similar to the equations of kinetic theory (Liouville equation in phase space): in front of the derivative along x , we find the velocity p/m . This simple form appears because there is no potential (free evolution). If the kinetic energy had higher-order terms not quadratic in p or if there is a potential, then derivatives with respect to p and higher derivatives appear in this equation.

The solution to Eq.(2.12) is simply

$$W(x, p, t) = W(x - pt/m, p, 0) \quad (2.13)$$

This translates the simple fact that the momentum distribution is constant in time (indeed, there is no force). The position distribution is “sheared” because, as in classical mechanics, particles with a positive velocity move to the right etc., as sketched in the following picture.



Now, if we measure the position distribution at time t , we get from the Wigner function (2.13) and the projection theorem:

$$\rho(x, t) = \int dp W(x - pt/m, p, 0) \quad (2.14)$$

which is a projection along a “tilted axis” in phase space.

Exercise 6. Write a program that plots the free evolution of the Wigner function for a superposition of two spatially separated wave packets.

2.1.3 Coherent state as reference states

We now switch to quantum optics and use the notation used there, in particular the dimensionless position and momentum operators that are built from the annihilation and creation operators

$$x = \frac{a + a^\dagger}{\sqrt{2}}, \quad p = \frac{a - a^\dagger}{\sqrt{2}i} \quad (2.15)$$

(No guarantee for the $1/\sqrt{2}$, different conventions exist.)

A key concept that we are going to use are the coherent (or Glauber) states, that we describe in detail in Sec.2.4.3. The basic properties of coherent states are the following.

Coherent states are “best suited” to represent classical fields, i.e., with a nonzero expectation value of the electric field (the position or momentum operators). This can be achieved by constructing an eigenstate of the annihilation operator:

$$a|\alpha\rangle = \alpha|\alpha\rangle \quad (2.16)$$

The eigenvalue α is complex because a is not a hermitean operator.

The expansion of a coherent state in terms of stationary states (with a fixed photon number n) is

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad (2.17)$$

This can be shown by working out the expansion of the eigenvalue Eq.(2.16) in the number state basis and solving a recurrence relation.

In the position representation, the formula (2.99) is not convenient at all (sum over Hermite polynomials), but the defining Eq.(2.16) becomes a relatively simple differential equation (we set $\alpha = (x_0 + ik)/\sqrt{2}$)

$$\begin{aligned} (\hat{x} + i\hat{p})|\alpha\rangle &= (x_0 + ik)|\alpha\rangle \\ (x + \partial_x)\psi_\alpha(x) &= (x_0 + ik)\psi_\alpha(x) \end{aligned} \quad (2.18)$$

It can be checked that the solution is a Gaussian wavepacket centred at $x = x_0$ and with average momentum k

$$\psi_\alpha(x) = N \exp[-\frac{1}{2}(x - x_0)^2 + ikx] \quad (2.19)$$

Note that the coordinates x, p used here are dimensionless, different from the section on the Wigner function.

The coherent states are “over complete” which can be checked from the overlap

$$\langle\alpha|\beta\rangle = e^{i\phi(\alpha,\beta)} \exp(-\frac{1}{2}|\alpha - \beta|^2) \quad (2.20)$$

where the phase $\phi(\alpha, \beta)$ is worked out in the exercises. This equation means that coherent states are not orthogonal (this would apply for the eigenstates of a hermitean operator). One can also show the relation

$$\int d^2\alpha |\alpha\rangle\langle\alpha| = \pi \mathbb{1} \quad (2.21)$$

where the integral $d^2\alpha = d(\text{Re } \alpha) d(\text{Im } \alpha)$ is taken over the real and imaginary parts of α . The factor π , larger than 1, illustrates that this is not a “completeness relation”, but that there are “too many” of coherent states.

2.1.4 The Q-function

A phase-space distribution that is related to the overlap with a Gaussian wave packet (a coherent state) is the *Husimi* or *Q-function* defined by

$$Q(\alpha) = \frac{1}{\pi} \langle\alpha|\hat{\rho}|\alpha\rangle \quad (2.22)$$

where α is the complex parameter of a coherent state. The prefactor $1/\pi$ ensures the normalisation: the integral of $Q(\alpha)$ over the entire α -plane is equal to the trace of $\hat{\rho}$, hence unity [see Eq.(2.21)]. Each density operator $\hat{\rho}$ defines a Q-function and more generally, the Q-function just provides an alternative characterization of the quantum state.

The Q-function has the following nice properties.

It is positive $Q(\alpha) \geq 0$ for all α and any density operator $\hat{\rho}$. This directly follows from $\hat{\rho}$ being a density operator and the coherent state $|\alpha\rangle$ being a normalizable Hilbert space vector.

For a pure coherent state, $\hat{\rho} = |\beta\rangle\langle\beta|$, the Q-function is a Gaussian centered at $\alpha = \beta$ and a spread of order unity, see Eq.(2.20).

Exercise. For a thermal state, $Q_T(\alpha)$ is a Gaussian centered at $\alpha = 0$ with a width of order $[\langle\hat{n}\rangle_T + 1]^{1/2}$.

How would the Q-function look for a number state? A first guess is a ‘ring’, since the photon number (or energy) is fixed and shows no fluctuations. This is not far from the precise answer that we have already calculated:

$$Q_n(\alpha) = \frac{1}{\pi} |\langle\alpha|n\rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{\pi n!} \quad (2.23)$$

where now the Poisson distribution has to be read as a function of α . It is manifestly isotropic, increases like a power law $|\alpha|^{2n}$ near the origin and decays in a Gaussian manner for large α . The maximum indeed occurs for $|\alpha|^2 \approx n$. The rim of this ‘volcano distribution’ becomes narrower and narrower as n increases.

2.1.5 The P-function

This function, also called Glauber-Sudarshan distribution, provides an expansion of the density operator in the basis of coherent states. There are two variants: the (‘simple’) P-function (Sudarshan, 1963)

$$\hat{\rho} = \int d^2\alpha P(\alpha) |\alpha\rangle\langle\alpha| \quad (2.24)$$

(the integration measure is again $d^2\alpha = d(\text{Re } \alpha) d(\text{Im } \alpha)$) and the ‘positive P-function’ (Glauber, 1963)

$$\hat{\rho} = \int d^2\alpha d^2\beta P(\alpha, \beta^*) |\alpha\rangle\langle\beta| \quad (2.25)$$

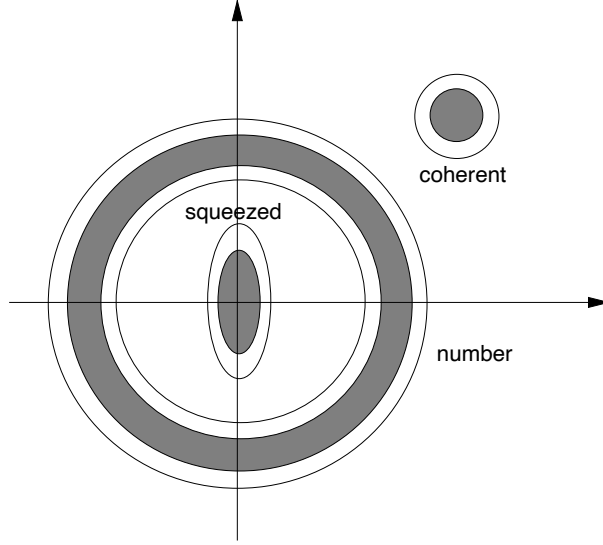


Figure 2.3: Q-functions for selected states of the radiation field.

It is actually surprising that many density operators (indeed, there are some exceptions) can be represented as a sum of projectors $|\alpha\rangle\langle\alpha|$ on coherent states. This is related to the coherent states being not orthogonal. The price to pay is also that the P-function $P(\alpha)$ can be a quite singular distribution, containing δ -functions and derivatives of δ -functions. The positive-P representation on the contrary always exists as a regular function $P(\alpha, \beta^*)$.

Formula by Mehta that gives P-function from density operator (taken from Lee 1991):

$$P(\alpha) = e^{|\alpha|^2} \int \frac{d^2\beta}{\pi} e^{\alpha\beta^* - \alpha^*\beta} \langle -\beta | \rho | \beta \rangle e^{|\beta|^2} \quad (2.26)$$

where the integral may not exist. This can be used to compute the Wigner function of a number state in terms of Laguerre polynomials.

Example: for a coherent state,

$$\hat{\rho} = |\beta\rangle\langle\beta| : \quad P_\beta(\alpha) = \delta(\alpha - \beta) \quad (2.27)$$

where the δ -function is defined with respect to the integration measure: $\delta(\alpha) = \delta(\text{Re } \alpha) \delta(\text{Im } \alpha)$.

It is easy to see, by taking the expectation value of Eq.(2.24) in a coherent state, that *the Q-function is a Gaussian convolution (Faltung) of the P-function*:

$$Q(\alpha) = \int \frac{d^2\beta}{\pi} P(\beta) \exp(-|\alpha - \beta|^2) \quad (2.28)$$

This explains why the Q-function behaves always ‘less singularly’ than the P-function.

2.1.6 Non-classical states

In the 1970s and 1980s, many people in the quantum optics community tried to define precisely what distinguishes the states of the quantum theory of radiation from classical optics. Long tradition in this area: *Arbeitsgruppe Nichtklassische Strahlung* led by Harry Paul at HU Berlin.

The coherent states had been introduced to have a “classical reference” – minimum uncertainty wave packets that allow for nonzero expectation values of position and momentum, while respecting the constraints of quantum mechanics (non-commuting operators). The Wigner function and the P-function provided two examples of more refined information: one could even compute a number telling how much a state could not be interpreted as a classical distribution.

In classical statistical mechanics, as you remember, the “state” of a system can be given as a probability distribution over phase space: each point (x, p) gives the (precisely known) phase-space coordinates of a particle, and the function $f(x, p)$ gives the number of particles near this point. (This description can be used for one particle or for an ensemble, e.g., for a gas.)

Singular P-function

One criterion for “non-classical light” took this idea to the P function (Sudarshan, 1963): *if the P-function $P(\alpha)$ is a regular positive function, then it describes a classical state of the radiation field*. Indeed, in that case, the density operator ρ can be understood as a “classical mixture” of projector onto coherent states, simply using Eq.(2.24) and the fact that coherent states are “classical”.

Conversely, if a P-function turns out to have singularities “more singular than a δ -function”, then the state is non-classical. Try to compute the P-function for a number state from Eq.(2.26): you will get a higher-order derivative of a δ -function $\delta(\alpha)$. Hence any number state with $n > 0$ is non-classical. But there are also other examples like squeezed states (see Sec.2.4.4).

Negative Wigner function

An alternative (and in fact, different) criterion is based on the regions in phase space where the Wigner function is negative. For example, the “size” of these regions has been taken as a quantitative measure of “non-classicality”. Related to this idea is the formula [Eq.(1) from Kenfack and Życzkowski (2004)]

$$\int dx dp \{ |W| - W \} = \int dx dp |W| - 1 \quad (2.29)$$

This yields zero for a Wigner function which is positive everywhere. The only examples that we have found so far of such Wigner functions are the ground state of the harmonic oscillator and the coherent states that have positive Wigner functions. In the cited paper, explicit numbers are computed for non-classical states like superpositions of wave packets, number states and squeezed number states.

The criterion (2.29) does not change when the Wigner function is displaced *en bloc*, as it happens when a displacement operator is applied to the quantum state (see exercise above). It also does not change when the state is “squeezed” – the Wigner then undergoes a shearing transformation (perhaps combined with a rotation), but does not change its values. (We shall see that this example demonstrates the difference in the “Wigner measure” of “non-classicality” compared to the singularity of the P-function.) The same is true for the time evolution in a harmonic potential because the Wigner function is rotated *en bloc*. This can be seen from the equation of motion

$$\partial_t W - \frac{p}{m} \partial_x W + m\omega^2 x \partial_p W = 0 \quad (2.30)$$

where ω is the oscillator frequency.

Exercise. Derive equation (2.30) from the Schrödinger equation in the position representation.

In the “natural units” used in quantum optics, this equation becomes

$$\partial_t W = \omega [p \partial_x - x \partial_p] W \quad (2.31)$$

where x and p are dimensionless coordinates for an oscillator with frequency ω . By using polar coordinates in phase space, $x = r \cos \varphi$ and

$p = r \sin \varphi$, one can show that the differential operator in square brackets is just $-\partial/\partial\varphi$, hence it simply generates a rotation of the Wigner function. Again, this does not change the size of the negative regions so that the non-classicality defined by Kenfack and Życzkowski remains invariant for the free evolution of a field mode. (Conclusion: to make a state “more non-classical”, one needs a potential higher than quadratic.)

2.1.7 Wigner function in quantum optics

In this section, we introduce the version of the Wigner function that is used in quantum optics: we shall work with the “natural dimensionless coordinates” x and p . They are related to the complex parameter of the coherent state according to $\alpha = (x+ip)/\sqrt{2}$ [see Eq.(2.18)]. It is convenient to re-scale the Wigner function when written in the complex variable α because of the integration measure $d^2\alpha = dx dp/2$ and the normalisation integral:

$$1 = \int dx dp W(x, p) = \int d^2\alpha W(\alpha), \quad W(\alpha) = 2W(x, p) \quad (2.32)$$

For the coherent state $|\beta\rangle$, we thus get from a previous exercise

$$W(\alpha) = \frac{e^{-2|\alpha-\beta|^2}}{\pi/2} \quad (2.33)$$

If we compare to the P-function which is $\delta^{(2)}(\alpha - \beta)$ for a coherent state and to the Q-function [Eqs.(2.20, 2.22)], we observe that their characteristic widths scale like $0 : 1/2 : 1$ for the P-, Wigner, and Q-function. In terms of (de)convolutions, the Wigner function is thus ‘mid-way’ between the P- and Q-functions.

Example. Thermal state

$$W_T(\alpha) = \frac{1}{\pi(\bar{n} + \frac{1}{2})} \exp\left(-\frac{|\alpha|^2}{\bar{n} + \frac{1}{2}}\right) \quad (2.34)$$

This is also an example where the width of the distribution, namely $\bar{n} + \frac{1}{2}$ changes, here it increases continuously with the temperature.

The different choices of phase space functions are related to the ordering of the operators a and a^\dagger . This can be seen from the construction of

Eq.(2.8) for the Wigner function. We introduced the expectation value of a displacement operator

$$\chi_W(z) = \langle \hat{D}(z) \rangle \quad (2.35)$$

and then Fourier-transformed it to construct the Wigner function. If we expand $\langle \hat{D}(z) \rangle$ in powers of z and z^* , then we get *symmetrically ordered products* of a and a^\dagger – simply because the complex numbers z and z^* commute. In the second order, for example:

$$\chi_W(z) = 1 + z\langle a^\dagger \rangle - z^*\langle a \rangle + \frac{z^2}{2}\langle a^{\dagger 2} \rangle + \frac{z^{*2}}{2}\langle a^2 \rangle - \frac{zz^*}{2}\langle a^\dagger a + aa^\dagger \rangle + \mathcal{O}(z^3) \quad (2.36)$$

In classical statistics, such a power series is called a “moment generating function”: an average like $\langle a^2 \rangle$ is called a second moment – and can be read off from the coefficient of the term z^{*2} . For a coherent state, $\chi_W(z)$ is a Gaussian: its Taylor expansion in powers of z and z^* then provides all symmetrically ordered moments.

The moment generating function we have constructed here contains only symmetric operator products. This implies that with the Wigner function, we can compute the average of such products by a simple replacement rule. Consider for example

$$\frac{1}{2}\langle aa^\dagger + a^\dagger a \rangle = \int d^2\alpha \frac{\alpha\alpha^* + \alpha^*\alpha}{2} W(\alpha) = \int d^2\alpha |\alpha|^2 W(\alpha) \quad (2.37)$$

Under the integral, we replace the operators a and a^\dagger by the complex numbers α and α^* , and then perform the integral. For the vacuum state, the variance $\frac{1}{2}\langle aa^\dagger + a^\dagger a \rangle$ is not zero because the Wigner function has a finite width around $\alpha = 0$ (a disk of “vacuum fluctuations”). The same formula does not apply for the P-function, because the integral on the right would give zero in the vacuum state: $P(\alpha)$ is a δ -peak centered at zero (“no fluctuations”). We shall see in Eq.(??) how the prescription has to be modified for the P-function.

As a second example, any moments of a quadrature can be computed ‘in the easy way’ with the Wigner function

$$\langle X^n \rangle = \int d^2\alpha \frac{(\alpha + \alpha^*)^n}{2^{n/2}} W(\alpha) \quad (2.38)$$

because in the expansion of the operator power X^n , a symmetric sum over products of the a and a^\dagger operators appears. In this way, one can also

compute the full distribution function $P(X_\theta = q)$ of any quadrature X_θ which is an experimentally measurable quantity (using homodyne detection, Sec. 2.5.2). The full Wigner function can be reconstructed by combining distribution functions for quadratures of several angles. This method is called “quantum state tomography” and provides the quantum optics version of the time-of-flight technique mentioned around Eq.(2.13).

W. Vogel (2000) from Rostock University has shown that whenever this technique gives a distribution function for quadratures that is “narrower” than the vacuum states, we are dealing with a non-classical state. This protocol has been put to use in an experiment by Parigi, Zavatta, and Bellini (2006) where non-classical states have been generated by adding exactly one photon to a thermal equilibrium state.

2.1.8 Characteristic functions

The moment generating function for the Wigner function $\chi_W(z)$ can be generalized to compute operator averages taken in a different order. For the P-function, for example, one defines

$$\chi(z) = \langle e^{za^\dagger} e^{-z^*a} \rangle \quad (2.39)$$

where the first exponential only generates powers of a^\dagger . The Taylor expansion in z and z^* preserves here the ordering of the operators: a^\dagger to the left, a to the right. In the vacuum state, we thus get $\chi(z) = 1$. (**Exercise.** And for a coherent state $|\beta\rangle$?)

Taking the Fourier transform, we get the P-function

$$P(\alpha) = \int \frac{d^2z}{\pi^2} e^{\alpha z^* - \alpha^* z} \chi(z) \quad (2.40)$$

and we can check this by noting that the quantity $\alpha z^* - \alpha^* z$ is purely imaginary and gives the natural phase factor for the Fourier transform. For the vacuum state, the integral then yields a two-dimensional δ -function, as it must (since this is a special case of a coherent state).

Non-classical states appear when the Fourier integral (2.40) does not converge. Already the vacuum state is a marginal case, but even worse results are obtained for number states, for example. (**Exercise.** Work out

$\chi(z)$ for that case and analyze the behaviour of the integrand in Eq.(2.40)) for large z .

An alternative viewpoint on normal order: we may look at variances of quadratures with the vacuum-value subtracted. It is easy to check for a quadrature X

$$(\Delta X)^2 - (\Delta X)_{\text{vac}}^2 = \langle : (X - \langle X \rangle)^2 : \rangle \quad (2.41)$$

where the colons $: \dots :$ denote the normal order of the operators. The correspondence between normally ordered products and their averages with respect to the P-function is expressed by the formula

$$(\Delta X)^2 - (\Delta X)_{\text{vac}}^2 = \langle : (X - \langle X \rangle)^2 : \rangle = \int d^2\alpha (x - \bar{x})^2 P(\alpha) \quad (2.42)$$

where $x = (\alpha + \alpha^*)/\sqrt{2}$ is the ‘classical version’ of the quadrature operator [see Eq.(4.49)] and $\bar{x} = \langle X \rangle$. Note the positive quantity $(x - \bar{x})^2$ under the integral over the P-function. As long as the P-distribution is regular, the result will be positive. Conversely, a squeezed state whose variance is below the vacuum value cannot have a regular P-function. According to this criterion for ‘being non-classical’, a squeezed state is indeed non-classical.³

Now comes a key insight: if we use the Baker–Campbell–Hausdorff formula, we can re-write the normally ordered operator average (2.39) and re-group the exponents to get

$$\chi(z) = e^{\frac{1}{2}|z|^2} \langle e^{za^\dagger - z^*a} \rangle = e^{\frac{1}{2}|z|^2} \chi_W(z) \quad (2.43)$$

We thus find, up to a Gaussian factor, the moment generating function of the Wigner distribution. Note the “inverted Gaussian” $e^{\frac{1}{2}|z|^2}$ that multiplies $\chi_W(z)$. This certainly makes the convergence of the Fourier integral more difficult for the P-function. This factor explains the difference in behaviour between the P- and the Wigner function.

The construction of Eq.(2.43) was generalized by Cahill to any real number s between +1 and -1

$$\chi_s(z) = \langle \hat{D}(z) \rangle e^{\frac{s}{2}|z|^2} \quad (2.44)$$

$s = 1 :$ P-function

³Note that the reference to the vacuum (ground) state implies that natural units for the X - and P -quadratures exist. This is why the ‘cheating way of squeezing’ with a simple re-scaling [see Eq.(2.124)] actually corresponds to a physical transformation.

$$\begin{aligned}
s = 0 & : && \text{Wigner function} \\
s = -1 & : && \text{Q-function}
\end{aligned}$$

This is sometimes called ‘*s*-ordering’. For $s = 0$, we get the generating function of the Wigner function, and for $s = -1$... the one for the Q-function which is $\langle e^{-z^* a} e^{z a^\dagger} \rangle$ (“anti-normal order”). The functions χ and χ_W giving the phase-space distributions by a Fourier transformation, we may apply the convolution theorem. Since χ and χ_W differ by a Gaussian factor, the Wigner function is the convolution of the P-function with a Gaussian, namely the Fourier transform of $e^{-\frac{1}{2}|z|^2}$. This gives the formula

$$W(\alpha) = \int \frac{d^2\beta}{\pi/2} e^{-2|\alpha-\beta|^2} P(\beta) \quad (2.45)$$

Compare with the corresponding expression (2.28) for the Q- and P-functions: the only difference is the width of the Gaussian. By playing with this width, Eq.(2.45) can be generalized to give a phase-space function for arbitrary s , sometimes called the Cahill–Glauber function $P(s; \alpha)$.

Now observe that for $s < 0$, the function $\chi_s(z) = e^{\frac{s}{2}|z|^2} \langle \hat{D}(z) \rangle$ decays more and more rapidly for large z . This means that its Fourier transform (for example the Q-functions) is “smoother” than the P-function. To adopt the language of image processing: By multiplying with $e^{\frac{s}{2}|z|^2}$ (for $s < 0$), large k -vectors are suppressed. But a picture with few high k -vectors is blurred, out of focus (*unscharf*). This is what happens under a Gaussian convolution. From a mathematical viewpoint, Wigner and Q exist as ordinary functions just because their Fourier transforms converge thanks to the Gaussian cutoff at large z . For the P-function, this does not happen, and one must take recourse to the singular functions of distribution theory.

This observation was used by Lee (1991) in order to introduce a measure “how much non-classical” a state is: this is the largest parameter $-1 \leq s^* \leq +1$ such that the back transform of $\chi_{s^*}(z)$ still exists (as a regular, positive distribution, but not more singular than a δ -function). If $s^* = 1$, then the P-function is a regular distribution itself, and the state is “classical” in the P-function sense. If $s^* = 0$, then the Wigner function is positive and the state is classical in the Wigner sense.

Example: squeezed states. The squeezed states now provide the following scenario. They are defined as Gaussian minimal uncertainty wave

packets such that there exist quadratures X and P with $\Delta X < \Delta P$ (while $\Delta X \Delta P = \frac{1}{2}$).

For a squeezed state, the Wigner function ($s = 0$) exists as a positive distribution. But there is a value $s^* > 0$ (approaching eventually the P-function), for which one can find a direction in phase space where the back transform does no longer exist: This threshold is related to one quadrature having a variance below the vacuum level, $\Delta X^2 \leq \frac{1}{2}$. Lee (1991) has found that the largest s defined in that way is related to the average number of thermal photons that one has to ‘mix’ with a non-classical state (in a beam splitter) in order to produce a ‘classical field’, i.e., to ‘spoil’ the quantum properties of the non-classical state. To describe such a ‘mixture’, one takes a kind of convolution of the P-function of the signal mode and the P-function of a thermal state.

2.2 Gaussian states and Gaussian operations

We now consider a class of states that is ‘closed’ under typical operations that appear in quantum optics. We provide the discussion in terms of the Wigner function because of the simple relation (2.44) for its moment generating function $\chi(z)$. (Recall that this is also called characteristic function.)

2.2.1 Gaussian states

...are those states whose Wigner function is a Gaussian. They are completely characterised by their average positions in phase space

$$\langle \alpha \rangle = \frac{\langle X + iP \rangle}{\sqrt{2}} \quad (2.46)$$

and their covariance matrix. To get a compact vectorial notation in phase space, we set $\vec{q} = (x, p)$ or use operators $Q_1 = X$ and $Q_2 = P$. The covariance matrix (for one mode) is then defined as the symmetrized correlation

$$C_{ij} = \frac{1}{2} \langle Q_i Q_j + Q_j Q_i \rangle - \langle Q_i \rangle \langle Q_j \rangle \quad (2.47)$$

This is a positive matrix, i.e., for all phase-space vectors \vec{q}

$$q_i C_{ij} q_j = \Delta(\vec{q} \cdot \vec{Q})^2 := \langle (\vec{q} \cdot \vec{Q})^2 \rangle - \langle \vec{q} \cdot \vec{Q} \rangle^2 \geq 0 \quad (2.48)$$

Note that the linear combination $\vec{q} \cdot \vec{Q}$ is a (real) quadrature operator if \vec{q} is suitably normalized.

Examples. Coherent states are Gaussian, squeezed states, too. Thermal states are also Gaussian, but number states are not.

2.2.2 Gaussian operations

... are those manipulations S on a quantum system that maintain the Gaussian character of its state. Most operations in linear optics are Gaussian:

- rotation = free time evolution
- squeezing = nonlinear optics operation with a coherent pump beam (see page 44)
- thermalize in contact with a bath
- displacements = mix with coherent state at a beam splitter
- ‘mix and trace out’ = mix with another beam at a beam splitter and discard the other output of the beamsplitter

Gaussian states for multiple modes are defined in a similar way, using mean values and covariance matrices. When the ‘other beam’ is ‘traced out’, one keeps only the sub-block of the covariance matrix that is relevant to the (transmitted signal) beam (= one output of the beamsplitter).

One can show that even a beamsplitter with a ‘vacuum input’ (no light, just vacuum fluctuations) mixes additional fluctuations into the signal beam. In this way, the squeezing can be reduced, for example. Conversely, a coherent state can be squeezed when a ‘squeezed vacuum’ is used at the other input of the beam splitter (see the Sec.2.5.2 on homodyne detection).

In the following, we focus on unitary Gaussian transformations. They come with a unitary operator S that transforms a state according to $|\psi\rangle \mapsto S|\psi\rangle$. To apply the transformation on a quadrature, we need to conjugate it (“sandwich”):

$$X \mapsto S^\dagger X S \quad (2.49)$$

The Gaussian transformations have the property that the resulting operators are linear combinations of the original operators X and P . Using the

index notation:

$$Q_i \mapsto S^\dagger Q_i S = \sum_j M_{ij} Q_j + \alpha_i \quad (2.50)$$

with a 2×2 matrix \mathbf{M} and a vector $\vec{\alpha}$. Under a unitary transformation, the new operators satisfy the same commutation relation as before. Writing $X' = S^\dagger X S$ and similar for P' , we have

$$[X', P'] = [S^\dagger X S, S^\dagger P S] = S^\dagger [X, P] S = i\hbar S^\dagger S = i\hbar \quad (2.51)$$

This will have consequences for the matrices \mathbf{M} that represent the action of the transformation S on the canonical observables.

2.2.3 Example: displacement operators

Work with characteristic Wigner function

$$\chi(z) = \langle D(z) \rangle \quad (2.52)$$

If the state transforms with an operator S , then we have

$$\chi(z) \mapsto \langle S^\dagger D(z) S \rangle = \langle \exp(z S^\dagger a^\dagger S - S^\dagger a S z^*) \rangle \quad (2.53)$$

For a displacement operator $S = D(\alpha)$, it can be shown that it “displaces” the canonical operators

$$S^\dagger a S = a + \alpha \quad (2.54)$$

Exercise. Prove this relation by setting $\alpha = \alpha' t$ and deriving a differential equation with respect to t .

We then get

$$\chi(z) \mapsto \langle \exp[z(a^\dagger + \alpha^*) - (a + \alpha)z^*] \rangle = \chi(z) \exp(z\alpha^* - \alpha z^*) \quad (2.55)$$

The exponent appearing here is a pure phase that we have seen a few times before. As for the Wigner function, we use the mapping to classical phase-space coordinates to provide a more geometric interpretation. Re-name the real and imaginary parts $z = (x + iy)/\sqrt{2}$ and introduce the vector $\vec{x} = (x, y)$. Similarly for α and $\vec{\alpha}$. Then,

$$z\alpha^* - \alpha z^* = i(y\alpha_x - x\alpha_y) \quad (2.56)$$

This is a bilinear and antisymmetric form on phase space for which we introduce the notation

$$\vec{x} \wedge \vec{\alpha} = x\alpha_p - y\alpha_x \quad (2.57)$$

This is called the ‘canonical (or symplectic) form’ in analytical mechanics. It gives the area (with a sign) of the rectangle spanned by the phase-space vectors \vec{x} and $\vec{\alpha}$. It can be computed from the three-dimensional vector product, and this is the reason for the ‘wedge’ notation \wedge which is associated to an antisymmetric product (It is used in French texts for the vector product and in mathematics for the outer product between differential forms.)

Coming back to the characteristic function $\chi(z)$ in Eq.(2.55), the displacement operator $D(\alpha)$ changes it by a phase factor that involves the symplectic form between the function argument \vec{x} and the displacement $\vec{\alpha}$:

$$\text{displacement: } \chi(\vec{x}) \mapsto \chi(\vec{x}) e^{-i\vec{x} \wedge \vec{\alpha}} \quad (2.58)$$

We now take the Fourier back transformation to the Wigner function. The integral to perform is

$$W(\vec{q}) = \int \frac{d^2 z}{\pi^2} e^{qz^* - q^* z} \chi(z) \quad (2.59)$$

We can re-write the exponential as a ‘symplectic Fourier phase’ according to $q = (q_x + iq_p)/\sqrt{2}$ and $\vec{q} = (q_x, q_p)$

$$qz^* - q^* z = i\vec{x} \wedge \vec{q} \quad (2.60)$$

The Fourier integral for the transformed Wigner function becomes

$$W(\vec{q}) \mapsto \int \frac{d^2 x}{2\pi^2} e^{i\vec{x} \wedge \vec{q}} \chi(\vec{x}) e^{-i\vec{x} \wedge \vec{\alpha}} \quad (2.61)$$

which is simply the Wigner function shifted in phase space:

$$W(\vec{q}) \mapsto \int \frac{d^2 x}{2\pi^2} e^{i\vec{x} \wedge (\vec{q} - \vec{\alpha})} \chi(\vec{x}) = W(\vec{q} - \vec{\alpha}) \quad (2.62)$$

See how the Wigner function is displaced ‘en bloc’ without changing its shape. One can in this way define ‘coherent squeezed’ or ‘coherent thermal’ states by starting from nontrivial initial states. In terms of the Wigner function, the operation is almost trivial, while expansions in number states etc. generate quite involved algebra.

2.2.4 Canonical transformations

We now consider a class of transformations S that are called ‘canonical (or symplectic)’. They are defined by the following properties: (i) the phase space operators are transformed in a linear way and (ii) this mapping preserves the symplectic form.

(i) The operator on the Hilbert space S is required to generate a linear map for the quadrature operators:

$$i = x, p : \quad S^\dagger Q_i S = \sum_j M_{ij} Q_j, \quad \text{or} \quad S^\dagger \vec{Q} S = \mathbf{M} \vec{Q} \quad (2.63)$$

(This corresponds in the notation used before to the special case $\vec{\alpha} = \vec{0}$.)

(ii) The matrix \mathbf{M} is such that the area spanned by two phase space vectors is unchanged (symplectic matrix or transformation)

$$(\mathbf{M}\vec{x}) \wedge (\mathbf{M}\vec{y}) = \mathbf{M}\vec{x} \wedge \mathbf{M}\vec{y} = \vec{x} \wedge \vec{y} \quad (2.64)$$

Often we shall drop the parentheses in the symplectic form. It turns out that this ensures that the commutator is unchanged when we pass from the operators \vec{Q} to $\mathbf{M}\vec{Q}$.

Exercise. Consider the operator $S = \exp(i\theta a^\dagger a)$ and show that its action on the canonical coordinates X and P corresponds to the Gaussian operation “rotation” listed above.

What happens to the characteristic function? Observe that the operators in the exponent can be written as a symplectic form⁴

$$\chi(\vec{x}) \mapsto \langle S^\dagger \exp(za^\dagger - az^*) S \rangle = \langle S^\dagger \exp(-i\vec{x} \wedge \vec{Q}) S \rangle \quad (2.65)$$

From the preceding properties (i) and (ii), this becomes

$$\langle \exp(-i\vec{x} \wedge S^\dagger \vec{Q} S) \rangle = \langle \exp(-i\vec{x} \wedge \mathbf{M}\vec{Q}) \rangle = \langle \exp(-i\mathbf{M}^{-1}\vec{x} \wedge \vec{Q}) \rangle \quad (2.66)$$

The mapping of the characteristic function is therefore simply a composition (*Hintereinanderausführung*):

$$\chi(\vec{x}) \mapsto \chi(\mathbf{M}^{-1}\vec{x}) \quad (2.67)$$

⁴If the symplectic form is defined for operators as $\vec{A} \wedge \vec{B} = A_x B_p - B_x A_p$, where the operator product is taken in opposite order for the two terms, then one even has $\vec{Q} \wedge \vec{Q} = i\hbar$ from the commutation relation between X and P .

with the inverse symplectic transformation.

Now to the Wigner function. The Fourier integral (2.59) is, using the substitution⁵ $\vec{x} = \mathbf{M}\vec{x}'$

$$W(\vec{q}) \mapsto \int \frac{d^2x}{2\pi^2} e^{i\vec{x} \wedge \vec{q}} \chi(\mathbf{M}^{-1}\vec{x}) = \int \frac{d^2x'}{2\pi^2} e^{i\mathbf{M}\vec{x}' \wedge \vec{q}} \chi(\vec{x}') \quad (2.68)$$

Using the same trick to shift the symplectic matrix onto the other factor in the wedge product [as in Eq.(2.64)], this becomes (again!)

$$W(\vec{q}) \mapsto \int \frac{d^2x'}{2\pi^2} e^{i\vec{x}' \wedge \mathbf{M}^{-1}\vec{q}} \chi(\vec{x}') = W(\mathbf{M}^{-1}\vec{q}) \quad (2.69)$$

This expression shows how the phase-space coordinates are first transformed under \mathbf{M}^{-1} and then the ('old') Wigner function is evaluated. This is squeezing or rotating the Wigner function, preserving its 'footprint' in terms of phase-space area. In particular the normalization is preserved:⁶

$$1 = \int d^2q W(\vec{q}) \quad (2.70)$$

Starting from the vacuum state, one can thus construct a whole set of Gaussian states. It requires a bit of Lie group analysis to check that any Gaussian state can be reached.

⁵Whose Jacobian is 1 because one can show that for symplectic matrices, $\det \mathbf{M} = 1$, since the preserved area is simply the determinant.

⁶No guarantee that all factors $2\pi^2$ in the integrals are correct.

2.3 In quantum optics, how to identify a single mode

The quantized light field can be in different states. We start here with a single mode of the field. This may be a oversimplification, but single-mode fields have become part of the experimental reality with the advent of high-quality optical cavities. These devices give an electromagnetic field whose amplitude, in the region between two well-reflecting mirrors, is much higher at some resonant frequencies. The ‘mode function’ is in this case not a plane wave, of course, but a standing wave. In the transverse directions, one often has a Gaussian profile. Around a cavity resonance, it is a frequent approximation to treat the full field as if it contained only a single mode. The coupling to other modes may be taken into account as a loss.

The electric field is given by

$$\mathbf{E}(\mathbf{x}, t) = E_1 \boldsymbol{\epsilon} \left(a(t) + a^\dagger(t) \right) \sin kz \quad (2.71)$$

where z is the coordinate along the cavity axis and $k = n_z \pi / L$. The factor E_1 can be called the ‘electric field per photon’. From a plane-wave expansion in a quantization volume V , E_1 is given by the prefactor $E_k = (\hbar \omega_k / 2 \epsilon_0 V)^{1/2}$. The corresponding ‘intensity’ is

$$I_{1\text{ph}} = \epsilon_0 c E_k^2 = \frac{\hbar \omega_k c}{2V}. \quad (2.72)$$

In a cavity, we can take for V the volume ‘filled’ by the mode. For a transverse mode size of 1 micrometer and a cavity length of 1 cm, we get $I_{1\text{ph}} \sim 10^3 \text{ mW/cm}^2$ which is not really small. The total power, however, is quite small: about 10^{-8} W . Note also that these numbers are based on very ‘tight’ (diffraction-limited) focussing — beams with larger cross-section have a smaller ‘field per photon’.

In the Heisenberg picture, the field operator evolves as

$$\mathbf{E}(\mathbf{x}, t) = E_1 \boldsymbol{\epsilon} \left(a e^{-i\omega t} + a^\dagger e^{i\omega t} \right) \sin kz \quad (2.73)$$

A combination of annihilation and creation operators like the one in parentheses is called a ‘quadrature’. Quadratures always come in pairs. One

can find a second quadrature variable by shifting the origin of time by one quarter period: $\propto -i a e^{-i\omega t} + i a^\dagger e^{i\omega t}$. This corresponds to the magnetic field [compare expansion of electric and magnetic field operators in QO I]. In analogy to the harmonic oscillator, one often uses the following quadrature variables

$$X = \frac{a + a^\dagger}{\sqrt{2}} \quad P = \frac{a - a^\dagger}{\sqrt{2}i} \quad (2.74)$$

or more generally

$$X_\theta = \frac{a e^{-i\theta} + a^\dagger e^{i\theta}}{\sqrt{2}} \quad (2.75)$$

with $X_0 = X$ and $X_{\pi/2} = P$.

The ground state of the field mode is called the ‘vacuum’ (no photon, i.e., no excitation present). It is found by looking for the state that is annihilated by the annihilation operator: $a|\text{vac}\rangle = 0$. Obviously, this is also an eigenstate of the photon number operator with zero photons: $|\text{vac}\rangle = |0\rangle$. In the vacuum state, the electric field is also zero on average, of course.

But there are fluctuations around this average, called ‘quantum noise’. In the vacuum state of the single mode (2.71), e.g., we get

$$\langle \mathbf{E}(\mathbf{x}, t)^2 \rangle_0 = E_1^2 \sin^2 kz \langle 0 | (a(t) + a^\dagger(t)) (a(t) + a^\dagger(t)) | 0 \rangle \quad (2.76)$$

and this combination of operators gives an average

$$\langle 0 | (a(t) + a^\dagger(t)) (a(t) + a^\dagger(t)) | 0 \rangle = \langle 0 | a(t) a^\dagger(t) | 0 \rangle = 1 \quad (2.77)$$

The ‘vacuum noise’ in our mode is thus given by the squared single photon field $E_1^2 \sin^2 kz$. Similarly, the other quadrature variable $a(t) - a^\dagger(t)$ shows a noise strength of unity. This is in accordance with Heisenberg’s indeterminacy relation, since

$$[a(t) + a^\dagger(t), a(t) - a^\dagger(t)] = -2. \quad (2.78)$$

2.4 Gallery of quantum states

We give in this section an overview of different states that have been of interest in quantum optics. The phase space distribution functions can be

used to represent these states graphically, see Fig. 2.3 below. In the subsections, we also give “protocols” how to prepare the states experimentally in quantum optics.

The following Table 2.1 lists typical states and a few properties. Note that these states apply to any physical system described by a harmonic oscillator, this is a larger class than just modes of the electromagnetic field.

name	person	notation	eigenstate of	$\langle \hat{n} \rangle$	preparation
number	Fock	$ n\rangle$	$\hat{n} = \hat{a}^\dagger \hat{a}$	n	micromaser (difficult!)
thermal	Boltzmann	ρ_T	?	$\bar{n}(T)$ (BE)	contact with thermal bath
coherent	Glauber	$ \alpha\rangle = \hat{D}(\alpha) 0\rangle$	\hat{a}	$ \alpha ^2$	classical source
squeezed	?	$ \xi\rangle = \hat{S}(\xi) 0\rangle$	$(\mu a - \nu a^\dagger) \xi\rangle = 0$	$ \nu ^2$	non-linear medium, parametric resonance

Table 2.1: Quantum states of a single mode (annihilation operator a). The operator $\hat{D}(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ is called displacement operator. Similarly, $\hat{S}(\xi) = \exp(\xi a^{\dagger 2} - \xi^* a^2)$ for the squeezing operator. Parametrization $\xi = (r/2)e^{i\phi}$ with $\mu = \cosh r$, $\nu = e^{i\phi} \sinh r$. (Check factor 1/2.)

A graphical representation is shown in Fig. 2.4 where the squeezed state is the ellipse around the origin. These qualitative graphs can be made more quantitative by calculating the Q-function of the different quantum states discussed so far (see Sec. 2.1.4).

2.4.1 Number (Fock) states

The simplest quantum states of the single mode field are given by the well-known stationary states of the harmonic oscillator. These quantum states are called ‘Fock states’ or ‘number states’ $|n\rangle$. They are eigenstates of the ‘photon number operator’

$$\hat{n} = a^\dagger a = aa^\dagger - 1 \quad (2.79)$$

Since the field energy is proportional to the photon number, the Fock states are also eigenstates of the field Hamiltonian. Hence they correspond to the standard stationary states in quantum mechanics.

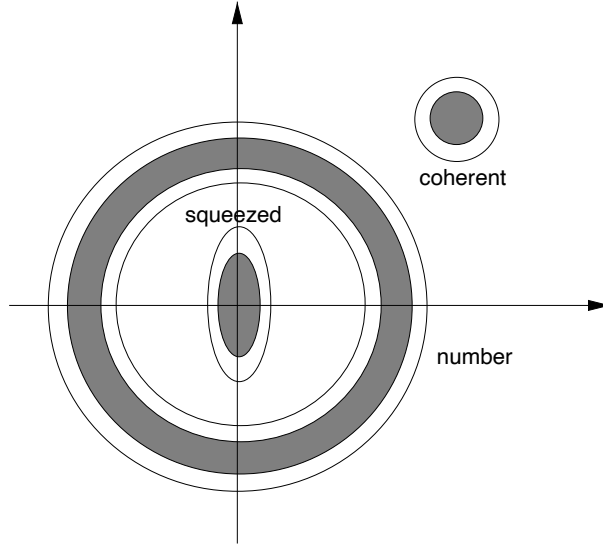


Figure 2.4: Quantum states of the radiation field, schematically represented in the phase space plane.

Number states are generated by applying the creation operator to the ground state (vacuum state) of the mode:

$$|n\rangle = \frac{1}{\sqrt{n!}}(a^\dagger)^n|0\rangle \quad (2.80)$$

The expectation value of the annihilation operator is zero in a number state:

$$\langle a \rangle_n = \langle n|a|n\rangle = \sqrt{n}\langle n|n-1\rangle = 0 \quad (2.81)$$

The same is true for the creation operator. It follows that the electric field average vanishes not only in the vacuum state, but in any Fock state:

$$\langle n|\mathbf{E}(\mathbf{x}, t)|n\rangle = 0 \quad (2.82)$$

Exercise. Compute the variances of the quadrature operators X_θ in an arbitrary number state $|n\rangle$.

The quantum numbers n give an intuitive interpretation to the creation and annihilation operators: they connect states whose photon numbers differ by one. In this sense, the ‘creation operator’ a^\dagger creates one photon since for example

$$\langle 1|a^\dagger|0\rangle = 1. \quad (2.83)$$

This matrix element plays an important role when one computed the probability amplitude that an excited atomic state emits a photon. For stimulated emission, one needs $\langle n+1|a^\dagger|n\rangle = \sqrt{n+1}$. Similarly, the ‘annihilator’ a destroys one photon:

$$\langle 0|a|1\rangle = 1. \quad (2.84)$$

This matrix element is needed to compute absorption, and in the general case, $\langle n-1|a|n\rangle = \sqrt{n}$.

- Experimentally, Fock (number) states are the most difficult to prepare. One has to avoid the loss of photons that makes the photon number uncertain. In addition, the preparation has to target precisely the photon number. One possible scheme works with a variant of the Jaynes-Cummings-Paul model: an excited two-level atom interacts with the single-mode cavity over a precisely tuned interaction time τ such that $g\sqrt{n+1}\tau$ is a multiple of 2π . As we have seen in QO I, the atom then performs a full Rabi cycle and ends again in the excited state. If the other conditions are well-chosen, the state $|n\rangle$ of the cavity can be a stable equilibrium state for this pumped system. Of course, one has to inject atoms regularly to compensate for the loss.

2.4.2 Thermal states

This class of field states is more general than the ‘pure’ states described before. Strictly speaking, they are not “states”, but density operators. The thermal state is the first example where one has to use both classical and quantum statistics, and this is achieved with the concept of *density operator* that combines the two.

Density operators

A *density operator* is a hermitean operator $\hat{\rho}$ on the Hilbert space \mathcal{H} of the quantum system under consideration, with the properties

- ρ is positive, i.e., $\langle \psi|\hat{\rho}|\psi\rangle \geq 0$ for all $\psi \in \mathcal{H}$
- ρ is a trace class operator, i.e., $\text{tr } \hat{\rho} = \sum_n \langle n|\hat{\rho}|n\rangle = 1$ where the vectors $|n\rangle$ form a basis of \mathcal{H} .

It is easy to see the inequality $0 \leq \langle \psi | \hat{\rho} | \psi \rangle \leq 1$ for a normalized state vector. Physically, this means that this the real number can be interpreted as a probability: it is the probability to find the system in the state $|\psi\rangle$ when performing a measurement.

The expectation value of an operator A is now given by the rule

$$\langle A \rangle_{\hat{\rho}} = \text{tr}(A\hat{\rho}) = \text{tr}(\hat{\rho}A) \quad (2.85)$$

where the order under the trace can be changed because of cyclic permutations.

In a sense, thermal quantum states are a natural generalization of classical thermodynamics to the quantum world. One uses stationary states, hence the number states we found first, and imposes Boltzmann statistics to describe the field at thermal equilibrium.

For the single field mode we are discussing here, stationary states are the number states $|n\rangle$; they occur with a classical probability proportional to the Boltzmann factor $e^{-n\hbar\omega/k_B T}$. The density operator is given by

$$\hat{\rho} = \frac{1}{Z} \exp[-\hbar\omega\hat{n}/k_B T] = \frac{1}{Z} \sum_{n=0}^{\infty} e^{-n\hbar\omega/k_B T} |n\rangle\langle n| \quad (2.86)$$

The normalization factor Z is found by requiring that the trace of this operator be unity:

$$Z = \text{tr} \left(\sum_{n=0}^{\infty} e^{-n\hbar\omega/k_B T} |n\rangle\langle n| \right) = \sum_{n=0}^{\infty} e^{-n\hbar\omega/k_B T} = \frac{1}{1 - e^{-\hbar\omega/k_B T}}, \quad (2.87)$$

where a geometric series has been summed. You know this sum from classical thermodynamics as ‘partition function’ (*Zustandssumme*). The normalized probabilities

$$p_n(T) = (1 - e^{-\hbar\omega/k_B T}) e^{-n\hbar\omega/k_B T} \quad (2.88)$$

are simply the classical probability that the stationary state $|n\rangle$ is realized in the canonical ensemble.

We note that the terms $|n\rangle\langle n|$ in the sum (2.86) are also density operators: they are obviously positive and have trace unity. (In fact, the trace boils down to the norm squared of the state $|n\rangle$.) The thermal density operator is thus a probability-weighted, convex sum of density operators.⁷ This

⁷One talks about a *convex sum* if all coefficients are real numbers between zero and one.

convex summation is, in general, an allowed linear operation on the space of density operators.

The density operators $|n\rangle\langle n|$ are special because they are made up of a single state. These quantum states are called *pure*. A formal definition:

- A density operator $\hat{\rho}$ describes a *pure state* if $\hat{\rho}^2 = \hat{\rho}$.

In mathematics, operators with this property are called *projectors*. This is also what is suggested by the Dirac notation $|\psi\rangle\langle\psi|$: this operator acts on the Hilbert space by first projecting onto the state $|\psi\rangle$ and then gives back a vector proportional to $|\psi\rangle$, just what happens in geometry for the projection onto a vector.

A thermal field mode

At optical frequencies and room temperature, the Boltzmann factor $\exp(-n\hbar\omega/k_B T)$ has a large negative argument for $n \geq 1$ so that the field is essentially at zero temperature. This is different for microwave radiation, e.g., or for star atmospheres.

Simple exercise: mean photon number. Let us apply the general rule (2.85):

$$\langle \hat{n} \rangle_T = \text{tr}(\hat{n} \hat{\rho}_T) = \frac{1}{Z} \sum_{n=0}^{\infty} \langle n | \hat{n} \exp(-\hbar\omega \hat{n} / k_B T) | n \rangle \quad (2.89)$$

The number operators and the Boltzmann ‘operator’ act on their eigenvectors, hence

$$\langle \hat{n} \rangle_T = \frac{1}{Z} \sum_{n=0}^{\infty} n \exp(-\hbar\omega n / k_B T) = \frac{1}{e^{\hbar\omega/k_B T} - 1}. \quad (2.90)$$

Exercise: photon number variance. Result:

$$(\Delta n)_T^2 = \frac{e^{\hbar\omega/k_B T}}{(e^{\hbar\omega/k_B T} - 1)^2} = \frac{1}{4 \sinh^2(\hbar\omega/2k_B T)}. \quad (2.91)$$

Discuss the limiting cases $\omega/T \rightarrow 0$ (‘hot’ or ‘classical’ limit) and $\omega/T \rightarrow \infty$ (‘cold’ or ‘quantum’ limit).

Electric field fluctuations in a single mode at finite temperature:

$$\langle \mathbf{E}^2(\mathbf{x}, t) \rangle_T = E_1^2 \sin^2 kz \langle a(t)a^\dagger(t) + a^\dagger(t)a(t) \rangle_T = E_1^2 \sin^2 kz (2\langle \hat{n} \rangle_T + 1) \quad (2.92)$$

they are enhanced by a factor $2\langle \hat{n} \rangle_T + 1 = \coth(\hbar\omega/2k_B T)$ compared to zero temperature.

Three remarks on the advantages of the density operator formalism:

- the traces that are required for expectation values can be taken in any basis. One can choose a basis adapted to the operator whose average one is interested in.
- Second, the presence of the density operator $\hat{\rho}$ under the trace ensures that the trace exists even if the operator A has ‘large matrix elements’ (like the photon number operator). Well, this is in fact just a restriction on the observables and states that are mathematically allowed. Thermal states have the advantage that the expectation values exist for a broad class of observables because the matrix elements of the density operator become rapidly small for large n .
- The third advantage of using a density operator approach is that it gives a suitable description of a quantum system whose dynamics is not completely known and can only be specified by probabilities. In that case, one formulates an equation of motion for the density matrix from the solution of which the averages of all interesting quantities can be calculated.

Preparation of a thermal state with rate equations

As an example of the last remark, we sketch here a ‘preparation scheme’ for a thermal state. We are going to use ‘rate equations’: differential equations for the diagonal elements $p_n(t) = \langle n | \hat{\rho}(t) | n \rangle$:

$$\frac{dp_n}{dt} = -\kappa n p_n + \kappa' n p_{n-1} - \kappa'(n+1)p_n + \kappa(n+1)p_{n+1} \quad (2.93)$$

The constants κ and κ' can be interpreted as transition rates between states: the transition $|n\rangle \rightarrow |n-1\rangle$ happens with the rate κn (this rate appears as

a negative term in \dot{p}_n and as a positive term in \dot{p}_{n-1}). This process can be interpreted physically as the loss of one of the n photons. This photon goes into a ‘heat bath’ or ‘environment’ and is absorbed there. Similarly, the system described by $\hat{\rho}$ can absorb one photon from the heat bath – this happens with a ‘Bose stimulation factor’ because for the transition $|n-1\rangle \rightarrow |n\rangle$, the rate is $\kappa'n$. (To be read off from the second and third terms in Eq.(2.93).) Even the vacuum state can absorb a photon, hence not $n-1$, but n appears here.

If one waits long enough, the density matrix (more precisely, its diagonal elements) relaxes into a steady state given by the equations of ‘detailed balance’

$$0 = -\kappa n p_n^{(\text{ss})} + \kappa' n p_{n-1}^{(\text{ss})} \quad (2.94)$$

This equation implies that $\dot{p}_n = 0$ in Eq.(2.93), but is slightly stronger. (One can probably show it by induction, starting from $n = 0$.) Eq.(3.35) gives a recurrence relation that links $p_n^{(\text{ss})}$ to $p_{n-1}^{(\text{ss})}$, whose solution is

$$p_n^{(\text{ss})} \sim \left(\frac{\kappa'}{\kappa} \right)^n =: e^{-n\hbar\omega/k_B T} \quad (2.95)$$

where we can identify the temperature T from the ratio of the rate constants κ'/κ . (One needs $\kappa' < \kappa$, otherwise no stable equilibrium state is found.) Of course, this definition of temperature is linked to assigning an energy $n\hbar\omega$ to the state $|n\rangle$.

Note the similarity of this approach to the photon statistics à la Scully & Lamb for the laser [QO I]. The difference here is that the dependence of the rates κn and $\kappa' n$ in the detailed balance relation (3.35) is in fact simpler because there is no saturation.

2.4.3 Coherent states

Definition and properties

The coherent state $|\alpha\rangle$ is an eigenstate of the annihilation operator:

$$a|\alpha\rangle = \alpha|\alpha\rangle \quad (2.96)$$

Since a is not an hermitean operator, α can be complex. In a coherent state, the average electric field is nonzero:

$$\langle \mathbf{E}(\mathbf{x}, t) \rangle_\alpha = E_1 \sin kz \langle \alpha | (a(t) + a^\dagger(t)) | \alpha \rangle = E_1 \sin kz (\alpha e^{-i\omega t} + \alpha^* e^{i\omega t}). \quad (2.97)$$

We have assumed the field in a coherent state of the initial annihilator a . This expression is the same that we have used in chapter 1 for a classical, monochromatic field. The magnetic field quadrature also has on average its classical value in a coherent state. Coherent states are thus very useful to represent laser fields. We see that α measures the electric field strength in units of the ‘single photon field’ E_1 . If we compute the average photon number in a coherent state, we get

$$\langle \hat{n} \rangle_\alpha = \langle a | a^\dagger a | \alpha \rangle = |\alpha|^2, \quad (2.98)$$

so that as an order of magnitude $\langle E \rangle \approx E_1 \langle \hat{n} \rangle^{1/2}$ (note the nonlinear dependence).

Coherent states are not stationary, but rotate in the complex α -plane: if $|\psi(0)\rangle = |\alpha\rangle$, then $|\psi(t)\rangle = |\alpha e^{-i\omega t}\rangle$. This can be shown using the expansion of a coherent state in terms of number states:

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \quad (2.99)$$

Note that number states with arbitrarily high photon numbers are present in a coherent state. More specifically, we can introduce the probability $p_n(\alpha)$ of finding n photons in a coherent state:

$$p_n(\alpha) = |\langle n | \alpha \rangle|^2 = e^{-|\alpha|^2} \frac{|\alpha|^{2n}}{n!} \quad (2.100)$$

which is a ‘Poisson distribution’ (the probability distribution of the sum of independent random bits). *Exercise:* compute the average photon number and its fluctuations (variance) in a coherent state:

$$\langle \hat{n} \rangle_\alpha = |\alpha|^2, \quad (\Delta n)_\alpha^2 = |\alpha|^2 \quad (2.101)$$

Note the important limiting case where the average photon number becomes large $|\alpha| \gg 1$. Then, the relative fluctuation of the photon number becomes small: $\Delta n / \langle n \rangle \sim 1/|\alpha| \ll 1$.

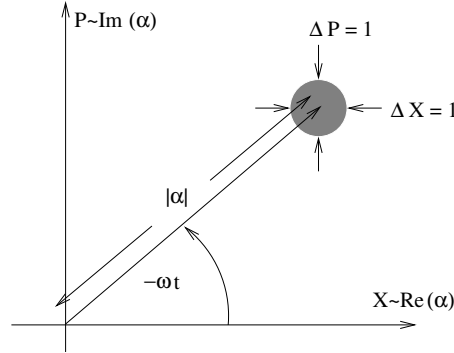


Figure 2.5: Representation of a coherent state in phase space.

The field quadratures also show quantum fluctuations around their classical average in a coherent state. This is inevitable because of the Heisenberg inequality. In the exercises, you are asked to show that these are equal to the quantum noise in the vacuum state (which is in fact a particular coherent state with $\alpha = 0$). This result can be displayed graphically in the complex α -plane by the sketch shown in fig. 2.5. We shall see that this plot gives the so-called Q-function (or Husimi function) of the state, see Eq.(2.22) below. This function provides a way to illustrate a quantum state by the analogy to the classical phase space. Note that since $a = (X + iP)/\sqrt{2}$, we may identify the α -plane with the classical phase space of a harmonic oscillator. The gray area in this sketch indicates values for the position and momentum quadratures that are probable outcomes of measurements. This representation is of course schematic since X and P cannot be measured simultaneously. We shall give it a precise meaning in section 2.1 where we show how coherent states can be used to expand any field state. (There are some subtleties related to the fact that they are not eigenstates of an hermitean operator.)

Finally, coherent states are not orthogonal. This is again a consequence of being the eigenstate of a non-hermitean operator. Let us calculate the overlap

$$\begin{aligned} \langle \alpha | \beta \rangle &= \sum_n e^{-|\alpha|^2/2 - |\beta|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^{*n} \beta^n}{n!} \\ &= \exp \left[-\frac{1}{2} (|\alpha|^2 + |\beta|^2 - \alpha^* \beta - \alpha \beta^*) \right] \end{aligned}$$

$$= \exp \left[-\frac{1}{2} |\alpha - \beta|^2 + i \operatorname{Im} \alpha^* \beta \right] \quad (2.102)$$

Here, we have split the complex overlap into its magnitude (a Gaussian with maximum at $\beta = \alpha$) and a phase factor. If we consider the Gaussian as a function of α , we get a peaked function in the phase space plane, with a typical width (the same in all directions) of the order of $\frac{1}{2}$ or 1.

Coherent states can be prepared by feeding the field mode with a “classical source”. This could be a classical oscillating dipole, as it happens in a so-called “free electron laser”. Or the field of a intense laser which is often approximated by a classical field. More details including the calculation of the time evolution operator for a classical source follow now.

Preparation: displacement operator

How is it possible to generate a coherent state physically? One possible answer is ‘never’ because to this end, one must be able to control the phase of the complex number α , or equivalently, the origin of time (recall the discussion before Eq.(2.99)). In practice, however, it is at least useful, if not necessary, to think ‘as if’ the phase of a light field were controlled, for example in a laser field. For an instructive discussion, see two papers by Klaus Mølmer (1997) where he talks about a ‘convenient fiction’. A physical example where it is plausible that the phase of a light field can be controlled is the ‘free electron laser’ where a beam of electrons is modulated in a controlled way (in a ‘wiggler’ element of an accelerator ring). The accelerated electrons are emitting photons that are injected into a laser cavity.

This example comes close to the following single-mode Hamiltonian

$$H = \hbar\omega \left(a^\dagger a + \frac{1}{2} \right) + i\hbar \left(e^{-i\omega_s t} g a^\dagger - e^{i\omega_s t} g^* a \right) \quad (2.103)$$

where the first term is the energy of our mode and the second term describes

- the coupling of a classical dipole oscillator at frequency ω_s with the field mode
- or the coupling of a classical current density $\mathbf{j}(x, t)$ with the vector potential of the mode (via the minimal coupling interaction).

We are going to see that *classical sources generate coherent states*.

In the interaction representation, the first term of the Hamiltonian (2.103) is transformed away and the exponentials are replaced by $e^{\pm i(\omega - \omega_s)t}$. If we choose resonant conditions, $\omega_s = \omega$, we thus get a Schrödinger equation with a time-independent Hamiltonian. The solution is easy (still in the interaction picture)

$$|\tilde{\psi}(t)\rangle = \exp[t(ga^\dagger - g^*a)]|\tilde{\psi}(0)\rangle = \hat{D}(gt)|\tilde{\psi}(0)\rangle \quad (2.104)$$

where $\hat{D}(\alpha)$ is the so-called *displacement operator*

$$\hat{D}(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) \quad (2.105)$$

Let us assume that the mode starts in the vacuum state, we thus find using the Baker-Campbell-Hausdorff identity⁸

$$|\psi(t)\rangle = \exp[t(ga^\dagger - g^*a)]|0\rangle = e^{-|g|^2 t^2/2} e^{gt a^\dagger} e^{-g^* t a}|0\rangle \quad (2.106)$$

Now the annihilation operator gives 0 when acting on the vacuum state, so that its exponential reduces to unity here. Expanding the exponential with the creation operator in a power series, we find

$$|\psi(t)\rangle = e^{-g^2 t^2/2} \sum_{n=0}^{\infty} \frac{(gt a^\dagger)^n}{n!} |0\rangle = |gt\rangle \quad (2.107)$$

This interaction thus generates a coherent state with amplitude $\alpha = gt$ that grows linearly in time. To obtain a stationary result, either the ‘oscillator amplitude’ g can be made time-dependent, or loss processes have to be added.

We have just shown that coherent states can be obtained by applying a ‘displacement operator’ to the vacuum state:

$$|\alpha\rangle = D(\alpha)|0\rangle \quad D(\alpha) = \exp\{\alpha a^\dagger - \alpha^* a\} \quad (2.108)$$

This unitary operator also displaces the creation and annihilation operators as follows (to prove by deriving a differential equation in the ‘Heisenberg picture’, setting $\alpha = gt$)

$$D^\dagger(\alpha) a D(\alpha) = a + \alpha \quad (2.109)$$

$$D^\dagger(\alpha) a^\dagger D(\alpha) = a^\dagger + \alpha^*. \quad (2.110)$$

⁸ If the commutator $[A, B]$ commutes with A and B : $e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B$.

This identity is useful to show that the field quadrature fluctuations ΔX and ΔP in a coherent state are those of the vacuum state.

The displacement operators provide a mapping from the complex numbers into unitary operators on the single-mode Hilbert space. Complex numbers can be added, and operators be applied sequentially. So how do the two operations compare? The answer lies in the equation

$$D(\alpha)D(\beta) = e^{i\text{Im}(\alpha\beta^*)}D(\alpha + \beta) \quad (2.111)$$

that can be easily proven with the Baker-Campbell-Hausdorff formula (footnote 8). If the phase factor were not there, this equation would make the mapping $\alpha \mapsto \hat{D}(\alpha)$ a *representation* (*Darstellung*) of the additive group in \mathbb{C} in the space of unitary operators $\mathcal{U}(\mathcal{H})$ over the (infinite-dimensional) Hilbert space \mathcal{H} of the single mode: either one applies the displacement operators one after the other (left-hand side) or one adds the complex numbers and applies a single displacement (right-hand side), one gets the same result.

Now, there is a phase factor, involving $\text{Im}(\alpha\beta^*)$. The mapping $\alpha \mapsto \hat{D}(\alpha)$ is then not a ('proper') representation, but only a *projective representation*. This must be so because the additive group in \mathbb{C} is finite-dimensional and commutative, while the unitary operators $D(\alpha)$ form a non-commutative group and are acting on an infinite-dimensional space. And more precisely, the *generators* of the two groups do not have the same algebra (a Lie algebra formed by their commutators). For the additive group and its action on \mathbb{C} itself, the generators can be taken as unit vectors parallel to the x and p axes. The addition of these vectors is, of course, commutative. For the 'image' formed by the $D(\alpha)$, acting on the Hilbert space of state vectors, the corresponding generators are (expand for small $\alpha = x + ip$ with real parameters x and p)

$$D(\alpha) \approx \mathbb{1} + x(a^\dagger - a) + ip(a^\dagger + a) \quad (2.112)$$

so we identify the generators $(a^\dagger - a)/i$ and $(a^\dagger + a)$ whose commutator is twice $i\mathbb{1}$. (One likes to choose hermitean generators, this explains the factors i . The commutator is hermitean after multiplication with i as well.) This means that the group structure is fundamentally different: the algebra spanned by the generators does not close, and a proper representation is

not possible. In fact, the additional phase factor that appears in the formula for the projective representation can be understood by enlarging the Lie algebra (and the group) to include also the unit operator.

To conclude, the phase factor appearing in Eq.(2.111) could be argued to have no physical significance: after all, changing a state vector by a ('global') phase does not change the quantum-mechanical predictions. But if a superposition can be constructed where the phase appears only in one term, then the phase becomes observable. A typical example is the 'geometric Berry phase'. We are not aware whether there is a link between this concept and the projective phase for the displacement operators.

2.4.4 Squeezed states

You should have got the feeling up to now that the quantized field essentially differs from a classical field by its ('quantum') fluctuations. So people have thought whether it is possible to reduce the quantum noise in a field quadrature to get something even 'more classical' – or having less noise. This can be achieved in part, to 50%, say. Of course, one cannot beat the Heisenberg inequality, and the reduced fluctuations in one quadrature have to be paid by enhanced noise in the other one.

Let us consider the following unitary operator

$$S(\xi) = \exp(\xi a^{\dagger 2} - \xi^* a^2) \quad (2.113)$$

Its action on the operators a and a^\dagger is the following linear transformation (also called Bogoliubov or squeezing transformation)

$$\begin{aligned} a &\mapsto S(\xi) a S^\dagger(\xi) = \mu a - \nu a^\dagger \\ a^\dagger &\mapsto S(\xi) a^\dagger S^\dagger(\xi) = \mu a^\dagger - \nu^* a \end{aligned} \quad (2.114)$$

where the squeezing parameters are

$$\mu = \cosh(2|\xi|), \quad \nu = e^{i\phi} \sinh(2|\xi|), \quad \phi = \arg(\xi) \quad (2.115)$$

To prove Eq.(2.114), one makes the replacement $\xi \mapsto \xi t$ and derives a differential equation with respect to the parameter t . (Mathematically: one studies the one-parameter family of squeezing operators $S(\xi t)$, a subgroup in the group of unitary transformations.)

The squeezed state $|\xi\rangle$ is now defined as the ‘vacuum state’ with respect to the transformed annihilation operator:

$$0 = S(\xi) a S^\dagger(\xi) |\xi\rangle = (\mu a - \nu a^\dagger) |\xi\rangle \quad (2.116)$$

This equation combined with the assumption that the vacuum state defined by $a|\text{vac}\rangle = 0$ is unique, gives $|\text{vac}\rangle = S^\dagger(\xi) |\xi\rangle$ after fixing a phase reference and therefore

$$|\xi\rangle = S(\xi) |\text{vac}\rangle \quad (2.117)$$

because S^\dagger is inverse to the unitary operator S . We thus get the squeezed state by applying the squeezing operator to the vacuum state.

One can also discuss more general cases, for example a squeezed coherent state $|\xi, \alpha\rangle = S(\xi) |\alpha\rangle = S(\xi) D(\alpha) |\text{vac}\rangle$. See the book by Vogel & al. (2001) for more details.

The squeezed state has a mean photon number

$$\langle \xi | a^\dagger a | \xi \rangle = \langle \text{vac} | S^\dagger(\xi) a^\dagger a S(\xi) | \text{vac} \rangle = \dots = |\nu|^2 \quad (2.118)$$

as can be shown by applying the transformation inverse to Eq.(2.114) (replace ξ by $-\xi$).

The photon number distribution reveals more interesting features. Consider first the case of a small squeezing parameter ξ . The expansion of Eq.(2.117) yields

$$|\xi\rangle = (1 + \xi a^{\dagger 2} - \xi^* a^2 + \dots) |\text{vac}\rangle = |\text{vac}\rangle + \sqrt{2} \xi |2\rangle + \dots \quad (2.119)$$

so that in addition to the ordinary vacuum, a state with a photon pair appears. This is a general feature: the squeezed (vacuum) state $|\xi\rangle$ contains pairs of photons, $|2\rangle, |4\rangle, \dots$. We shall see below that this can be interpreted as the result of a nonlinear process where a “pump photon” (of blue color, say) is “down-converted” into a pair of red photons. The unusual feature of this “photon pair state” is that the pair appears in a superposition with the vacuum state, with a relative phase fixed by the complex squeezing parameter ξ .

The expansion of the ‘squeezed vacuum’ $S(\xi)|0\rangle$ in the Fock (number state) basis gives for even photon numbers the amplitudes

$$c_{2m} = \frac{(2m-1)!!}{\sqrt{(2m)!}} e^{im\phi} \frac{\tanh^m(2|\xi|)}{\cosh^{1/2}(2|\xi|)} \quad (2.120)$$

where ϕ is again the phase of ξ , and $n!!$ is the product $n(n-2)\cdots$ of all positive numbers with the same parity up to n .

The mean value of the complex field amplitude is zero in the squeezed state, as a calculation similar to Eq.(2.118) easily shows: $\langle \xi | a | \xi \rangle = 0$. In the phase-space plane introduced in Fig. 2.5, the squeezed state $|\xi\rangle$ would therefore be represented by a “blob” centered at zero.

The “squeezing” becomes apparent if one asks for the quantum fluctuations around the mean value. Let us introduce the general quadrature operator

$$X_\theta = \frac{a e^{-i\theta} + a^\dagger e^{i\theta}}{\sqrt{2}} \quad (2.121)$$

The familiar position and momentum quadratures X , P correspond to phases $\theta = 0, \pi/2$. But more generally, two ‘orthogonal’ quadratures are given by X_θ and $X_{\theta+\pi/2}$. The squeezed state now has fluctuations around the vacuum state such that *one quadrature component has quantum noise below the Heisenberg limit* $1/2$. A straightforward calculation gives the following quadrature uncertainty

$$\langle \xi | \Delta X_\theta^2 | \xi \rangle = \frac{|\mu + \nu e^{-2i\theta}|^2}{2} \quad (2.122)$$

If we take $2\theta = \phi$ (the phase of the squeezing parameter), we have $\mu + \nu e^{-2i\theta} = \cosh(2|\xi|) + \sinh(2|\xi|) = e^{+2|\xi|}$ which becomes exponentially large as the magnitude of ξ increases. For the orthogonal quadrature, one finds an exponential reduction of the fluctuations:

$$\Delta X_{\phi/2}^2 = \frac{e^{+2|\xi|}}{2}, \quad \Delta X_{(\phi+\pi)/2}^2 = \frac{e^{-2|\xi|}}{2}. \quad (2.123)$$

This is the hallmark of a squeezed state. Note that the uncertainty product is unchanged: this could have been expected as $|\xi\rangle$ remains a pure state.

Preparation of a squeezed state

How can one prepare a squeezed state? The “cheating way of it” is just a re-scaling of the position and momentum quadratures:

$$X' = \eta X, \quad P' = \eta^{-1} P \quad (2.124)$$

This generates operators X' and P' that obey the same commutation relations. However, the energy of the field mode will not be proportional to $a'^{\dagger}a' \sim X'^2 + P'^2$, but involve terms of the form $(a')^2$ and $(a'^{\dagger})^2$. So the “ground state” $|\psi\rangle$ defined by $a'|\psi\rangle = 0$ will not be a stationary state of this Hamiltonian. This example illustrates, however, that (i) squeezed states evolve in time and are not stationary and (ii) that the quadratic terms $(a')^2$ and $(a'^{\dagger})^2$ play a key role.

The second way is to find a way to add these terms to the Hamiltonian. This can be done with a nonlinear medium. The ‘squeezing’ operator (2.113) can be realized with the interaction Hamiltonian

$$H_{\text{int}} = i\hbar \left(g e^{-2i\omega t} a'^{\dagger 2} - g^* e^{2i\omega t} a'^2 \right) \quad (2.125)$$

with the squeezing parameter given by $\xi = \int dt g(t)$. This interaction occurs in nonlinear optics. To get a qualitative understanding, imagine a medium with a field-dependent dielectric constant ($\chi^{(2)}$ nonlinearity). This is usually forbidden for symmetry reasons, but it happens in some special cases. In the electromagnetic energy density, one has

$$u = \frac{\varepsilon(|\mathbf{E}|)}{2} \mathbf{E}^2 + \frac{1}{2\mu_0} \mathbf{B}^2 \quad (2.126)$$

where the linearization

$$\varepsilon(|\mathbf{E}|) = \varepsilon_0 (1 + n_2 |\mathbf{E}|^2) \approx \varepsilon_0 (1 + 2n_2 |\mathbf{E}|^2)$$

is often appropriate. In the quantum picture, this gives a contribution to the Hamiltonian with a term of third order in the field:

$$H_3 = \varepsilon_0 n_2 \int_V d^3x |\mathbf{E}(\mathbf{x}, t)|^3 \quad (2.127)$$

Let us now pick out two spatial modes of the field and put one of it into a coherent state $|\alpha e^{-i\omega_p t}\rangle$ with a ‘large’ amplitude $|\alpha| \gg 1$. The index ‘p’ is for ‘pump field’. Let us call the other mode (the ‘quantum’ one) the ‘signal’. The electric field is then

$$\mathbf{E}(\mathbf{x}, t) = E_p a_p \boldsymbol{\varepsilon}_p e^{-i(\omega_p t - \mathbf{k}_p \cdot \mathbf{x})} + E_1 \boldsymbol{\varepsilon}_1(t) e^{i\mathbf{k} \cdot \mathbf{x}} + \text{h.c.} \quad (2.128)$$

The interaction Hamiltonian thus generates cross terms of the form⁹

$$H_{\text{int}} = \dots + \hbar \left(g e^{-i\omega_p t} a_p a^\dagger{}^2 + g^* e^{i\omega_p t} a_p^\dagger a^2 \right) \quad (2.129)$$

$$\hbar g = 3\varepsilon_0 n_2 E_p E_1 \boldsymbol{\varepsilon}_p \cdot \boldsymbol{\varepsilon}^* \int_V d^3x e^{i(\mathbf{k}_p - 2\mathbf{k}) \cdot \mathbf{x}} \quad (2.130)$$

One often ignores the quantum fluctuations of the pump mode and replaces its annihilation operator a_p by the coherent state amplitude α . The interaction (2.129) then looks quite like our model Hamiltonian (2.125).

The nonlinear squeezing parameter $g\alpha$ is nonzero when the pump and signal modes are ‘phase matched’, i.e., $\mathbf{k}_p = 2\mathbf{k}$. For collinear modes, this is achieved by taking $\omega_p = 2\omega$. The spatial integral actually runs only over the region where the nonlinear index n_2 is different from zero. We also see from (2.129) that one ‘pump photon’ with energy $\hbar\omega_p = 2\hbar\omega$ can ‘decay’ into a pair of signal photons. We already anticipated this behaviour in the number state expansion (2.119).

We finally get a time-independent Hamiltonian by assuming that the pump mode is in a coherent state, $a_p \mapsto \alpha_p$ and by going into a rotating frame at half the pump frequency, $a(t) = e^{-i\omega_p t/2} \tilde{a}(t)$. If one works in addition at exact resonance, the time evolution operator is $U(t) = S(\xi)$ with $\xi = g\alpha_p t$. In practice, one does not get infinite squeezing as $t \rightarrow \infty$ because of damping.

Two-mode squeezing

What we have seen so far is “one-mode squeezing”. The squeezed state can be used to create non-classical correlations between two bright beams.

Consider the output $a_{1,2} = (a \pm b)/\sqrt{2}$ of a balanced beamsplitter with squeezed vacuum state in mode a . This gives for suitable position and momentum quadratures the uncertainty product

$$\Delta(X_1 - X_2)\Delta(P_1 + P_2) < 1 \quad (2.131)$$

because the variance of the difference, $\Delta(X_1 - X_2)$, is just related to the squeezed variance $\Delta X < 1/\sqrt{2}$ of the input mode a . The other variable $P_1 + P_2$ has a variance related to the state of input mode b , it can be brought

⁹We are actually cheating with the polarization vector $\boldsymbol{\varepsilon}$. An accurate description replaces n_2 by a third-rank tensor that produces a scalar out of three vectors.

to a minimum uncertainty of order 1 with a coherent state in mode b . The inequality (2.154) is not inconsistent with the Heisenberg relations because the sum $P_1 + P_2$ and the difference $X_1 - X_2$ are commuting operators.

In other words, Eq.(2.154) tells us that the combination “squeezed vacuum + coherent state” sent onto a beam splitter provides two beams whose X -quadratures are correlated better than what is allowed by the standard vacuum fluctuations (or the fluctuations around a coherent = quasi-classical state). This is the criterion for a non-classical correlation.

Einstein, Podolski, and Rosen (1935) or “EPR” have discussed this arrangement in a slightly different form and came to the conclusion that quantum mechanics must be an incomplete theory. They mixed up, however, that the correlations we have here do not require some “instantaneous action at a distance” between the systems A and B (the two output beams after the beam splitter). Nonlocal correlations of this kind already appear in classical physics: hide a red and a blue ball in two boxes, move one box to the moon and open it. You know immediately the color of the other box, wherever it is. This correlation cannot be used to transmit information, however.

Unitary operator that generates two-mode squeezing:

$$S_{ab}(\xi) = \exp(\xi a^\dagger b^\dagger - \xi^* ab) \quad (2.132)$$

Exercise: check with single-mode squeezer (2.113) and beam splitter transformation (2.146). Appears in many different situations:

- non-degenerate nonlinear media (production of correlated photon pairs)
- normal modes of a degenerate, weakly interacting Bose gas (Bogoliubov quasi-particles)
- quantum field theory in classical background fields (Klein paradox, Hawking radiation, Unruh radiation), leading to “unstable vacuum states”

2.5 Quantum optics of the beamsplitter

recall scattering theory

transformation rules for mode operators, for quantum states

split a single photon (generate entanglement)

two-photon interference: Hong–Ou–Mandel experiment

homodyne measurement (local oscillator)

More details on multi-mode quantum fields can be found in Sec. 2.7.

2.5.1 State transformation

A beamsplitter is the most simple way to mix two modes, see Figure 2.6. From classical electrodynamics, one gets the following amplitudes for the

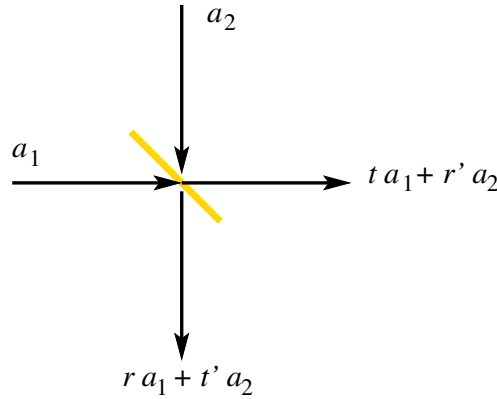


Figure 2.6: Mixing of two modes by a beam splitter.

outgoing modes:

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^{\text{in}} \mapsto \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^{\text{out}} = \begin{pmatrix} t & r \\ r' & t' \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}^{\text{in}}. \quad (2.133)$$

The recipe for quantization is now: ‘replace the classical amplitudes by annihilation operators’. If the outgoing modes are still to be useful for the quantum theory, they have to satisfy the commutation relations:

$$[a_i(\text{out}), a_j^\dagger(\text{out})] = \delta_{ij}. \quad (2.134)$$

These conditions give constraints on the reflection and transmission amplitudes, for example $|t'|^2 + |r'|^2 = 1$. Note that this is *not* identical

to energy conservation for the incoming mode $a_1(\text{in})$ [that would read $|t|^2 + |r'|^2 = 1$]. But a sufficient condition is that the classical ‘reciprocity relation’ (*Umkehrung des Strahlengangs*) holds: $t = t'$.

We are now looking for a unitary operator S [the S-matrix] that implements this beamsplitter transformation in the following sense:

$$a_i(\text{out}) = S^\dagger a_i(\text{in}) S, \quad i = 1, 2 \quad (2.135)$$

From this operator, we can also compute the transformation of the states: $|\text{out}\rangle = S|\text{in}\rangle$. Let us start from the general linear transformation

$$a_i \mapsto A_i = M_{ij} a_j \quad \text{or} \quad \vec{a} \mapsto \vec{A} = \mathbf{M} \vec{a} \quad (2.136)$$

where we have introduced matrix and vector notation. For the unitary transformation, we make the *Ansatz* (summation over j, k)

$$S(\theta) = \exp(i\theta B_{jk} a_j^\dagger a_k) \quad (2.137)$$

with B_{jk} a hermitean matrix (ensuring unitarity). The action of this unitary on the photon mode operators is now required to reduce to

$$a_i \mapsto A_i(\theta) \equiv S^\dagger(\theta) a_i S(\theta) \stackrel{!}{=} M_{ij} a_j. \quad (2.138)$$

Such an operation is called ‘conjugation with S ’. We compute it with a trick using a differential equation:

$$\frac{d}{d\theta} A_i(\theta) = -i B_{jk} S^\dagger(\theta) [a_j^\dagger a_k, a_i] S(\theta) \quad (2.139)$$

$$= -i B_{jk} S^\dagger(\theta) (-\delta_{ij} a_k) S(\theta) \quad (2.140)$$

$$= i B_{ik} A_k(\theta). \quad (2.141)$$

This is a system of linear differential equations with constant coefficients, so that we get as solution

$$\vec{A}(\theta) = \exp(i\theta \mathbf{B}) \vec{A}(0) = \exp(i\theta \mathbf{B}) \vec{a}. \quad (2.142)$$

We thus conclude that the matrix \mathbf{B} is fixed by

$$\mathbf{M} = \exp(i\theta \mathbf{B}). \quad (2.143)$$

If the transformation M is part of a continuous group and depends on θ as a parameter, we can expand it around unity. Doing the same for the matrix exponential, we get

$$M \approx \mathbb{1} + i\theta B + \dots$$

Here, B is called the *generator* of the set of matrices $M = M(\theta)$. The unitary transformation is thus determined via the same generator B .

For the two-mode beam splitter, an admissible transformation is given by

$$M = \begin{pmatrix} t & r \\ r' & t' \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}. \quad (2.144)$$

Expanding for small θ , the generator is

$$B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \sigma_2 \quad (2.145)$$

so that the corresponding unitary operator reads

$$S(\theta) = \exp [i\theta(-ia_1^\dagger a_2 + ia_2^\dagger a_1)] = \exp [\theta(a_1^\dagger a_2 - a_2^\dagger a_1)]. \quad (2.146)$$

Note that indeed, one has the identity

$$\exp(i\theta\sigma_2) = \cos \theta + i\sigma_2 \sin \theta = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \quad (2.147)$$

Example: splitting a single photon state

What is the state of the two-mode system if one photon is incident in mode 1 on the beam splitter? Initial state $|\text{in}\rangle = |1, 0\rangle = a_1^\dagger |\text{vac}\rangle$. The final state is then, using Eq.(2.146) for small θ

$$\begin{aligned} |\text{out}\rangle &= S|1, 0\rangle \approx |1, 0\rangle + \theta(a_1^\dagger a_2 - a_2^\dagger a_1)|1, 0\rangle \\ &= |1, 0\rangle - \theta|0, 1\rangle. \end{aligned} \quad (2.148)$$

For finite θ , the higher powers also contribute. The calculation gets easy with the beam splitter transformation of the creation operators.

$$\begin{aligned} |\text{out}\rangle &= Sa_1^\dagger |\text{vac}\rangle \\ &\stackrel{(1)}{=} Sa_1^\dagger S^\dagger |\text{vac}\rangle \\ &\stackrel{(2)}{=} (a_1^\dagger \cos \theta - a_2^\dagger \sin \theta) |\text{vac}\rangle \\ &= \cos \theta |1, 0\rangle - \sin \theta |0, 1\rangle \end{aligned} \quad (2.149)$$

In step (1), we have used that the unitary operator leaves the vacuum state unchanged. (This is because we have written the exponent in normal order.) In step (2), we have used that S implements the transformation inverse to S^\dagger (unitarity). Re-introducing the transmission amplitudes, we find

$$|1, 0\rangle \mapsto t|1, 0\rangle + r|0, 1\rangle \quad (2.150)$$

so that the probability amplitudes to find the photon in either output mode correspond exactly, for this incident one-photon state, to the classical transmission and reflection amplitudes.

It is quite complicated to show in the same way the following property of a ‘bi-coherent state’

$$S|\alpha, \beta\rangle = |\alpha', \beta'\rangle, \quad \begin{pmatrix} \alpha' \\ \beta' \end{pmatrix} = M \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \quad (2.151)$$

that remains bi-coherent after the beam splitter. But the proof is quite simple with the unitary transformation of the mode operators.

Example: splitting a two-photon state (Hong, Ou, Mandel)

Two-photon states do not behave as ‘intuitively’. Let us consider two single-photon states incident on the same beam splitter as before, $|\text{in}\rangle = |1, 1\rangle$. Then, by the same trick,

$$\begin{aligned} |\text{out}\rangle &= S|\text{in}\rangle = S a_1^\dagger S^\dagger S a_2^\dagger S^\dagger |0, 0\rangle \\ &= (a_1^\dagger \cos \theta - a_2^\dagger \sin \theta)(a_2^\dagger \cos \theta + a_1^\dagger \sin \theta) |\text{vac}\rangle \\ &= (|2, 0\rangle - |0, 2\rangle) \frac{\sin 2\theta}{2} + |1, 1\rangle \cos 2\theta \end{aligned} \quad (2.152)$$

Hence, for a 50/50 beam splitter ($\cos \theta = \sin \theta$ or $\theta = 45^\circ$), the last term cancels and the photons are transmitted in ‘bunches’: they come out together at either output port. There are zero ‘coincidences’ of one photon in port a_1' and the other in a_2' . This is due to a destructive interference between two indistinguishable histories for the two photons from source to detector – this is called the ‘Hong-Ou-Mandel dip’. The dip in the coincidence signal can be observed by tuning a parameter (like a delay time) that makes the two photons (in)distinguishable.

2.5.2 Homodyne detection

Introduce coherent state $|\beta\rangle$, simplest model for an intense laser beam.

Discuss output operators $a \pm \beta$ after a beam splitter: “mixing” of signal with “local oscillator” (= laser beam). The quadratures X_θ appear in the “beating” (interference) when a signal mode $a \mapsto a + \beta$ is mixed on a beam splitter with a large-amplitude coherent state $|\beta\rangle$ (“local oscillator”, “reference beam”). The quadrature phase can be chosen from the phase of β , in other words, the quadratures of a are measured relative to the phase of the local oscillator. (“Only relative phases are measurable.”)

Picture of quadratures in phase space plane for different states for signal mode a : vacuum state, number state, coherent state.

2.5.3 Photodetection

(see QO I lecture from WS 2019/20)

square-law detector signal (Glauber, 1960s)

$$I(t) = \langle \hat{I}(t) \rangle = \langle \hat{\mathcal{E}}^\dagger(t) \hat{\mathcal{E}}(t) \rangle \quad (2.153)$$

where the operator $\hat{\mathcal{E}}(t)$ gives the *positive frequency component* of the electric field, evaluated at the detector position and projected onto a polarization vector that reaches the detector. The positive frequency component of a free field operator contains all annihilation operators a_κ . These evolve in time as $e^{-i\omega_\kappa t}$, this motivates the name “positive frequency”. The conjugate operator $\hat{\mathcal{E}}^\dagger(t)$ is called the negative frequency component. Sometimes the notation $\hat{E}^{(-)}(t)\hat{E}^{(+)}(t)$ for the intensity operator is used.

Key feature: detector signal is nonzero only when photons are present, not in the vacuum state.

To remember: this is a “slow detector” – the derivation makes use of time-dependent perturbation theory and to get a sizable signal, one has to “wait” for many optical periods (to create a free electron, for example). This is the technical reason why the mixed product of negative/positive frequency operators appears and not the ordinary electric energy density, for example. Hence: Glauber’s theory does not work for “very fast” detectors (on the fs scale in the visible).

2.6 Entanglement and correlations

(Material not covered in SS 20.)

Entanglement is a property of two observables A and B or of two subsystems described by A and B . The two subsystems are called *entangled* when A and B show “non-classical correlations”, i.e., correlations that cannot be explained by classical statistics.

This formulation is similar to the negative (or singular) values of certain quasi-probabilities. When these cannot be interpreted as classical probabilities, we encounter non-classical states. The examples discussed below illustrate for example that squeezing in one mode can be used to entangle two modes – the “non-classicality” of one input state is a “resource” that provides “entangling power”.

2.6.1 EPR correlations

Section appears already earlier, in Sec. 2.4.4.

Output $a_{1,2} = (a \pm b)/\sqrt{2}$ of a balanced beamsplitter with squeezed vacuum state in mode a . This gives for suitable position and momentum uncertainties

$$\Delta(X_1 - X_2)\Delta(P_1 + P_2) < 1 \quad (2.154)$$

because the variance of the difference, $\Delta(X_1 - X_2)$, is just related to the squeezed variance $\Delta X < 1/\sqrt{2}$ of the input mode a . The other variable $P_1 + P_2$ has a variance related to the state of input mode b , it can be brought to a minimum uncertainty of order 1 with a coherent state. The inequality (2.154) is not inconsistent with the Heisenberg relations because the sum $P_1 + P_2$ and the difference $X_1 - X_2$ are commuting operators.

In other words, Eq.(2.154) tells us that the combination “squeezed vacuum + coherent state” sent onto a beam splitter provides two beams whose X -quadratures are correlated better than what is allowed by the standard vacuum fluctuations (or the fluctuations around a coherent = quasi-classical state). This is the criterion for a non-classical correlation.

Einstein, Podolski, and Rosen (1935) or “EPR” have discussed this arrangement in a slightly different form and came to the conclusion that quantum mechanics must be an incomplete theory. They mixed up, however, that the correlations we have here do not require some “instantaneous action at a distance” between the systems A and B (the two output beams after the beam splitter). Nonlocal correlations of this kind already appear in classical physics: hide a red and a blue ball in two boxes, move one box to the moon and open it. You know immediately the color of the other box, wherever it is. This correlation cannot be used to transmit information, however.

2.6.2 Bell correlations

The reasoning of EPR has been made more precise by John Bell (2001) who invented a systematic way of deriving inequalities (upper limits) to correlations between observables A and B . The “classical” assumption is that these take definite values (those that appear as outcomes of single measurements), but determined by some other “hidden variables” that obey classical statistics. If these “hidden variable theories” are formulated in a non-local way, any quantum correlation can be reproduced. But this would require assumptions that are not natural from the “local” viewpoint that has become familiar to us from relativity. An example of a local hidden variable theory provides an upper limit to spin correlations measured on two two-level systems with spin operators $\sigma \otimes \mathbb{1}$ (system A) and $\mathbb{1} \otimes \sigma$ (system B). More precisely, let us take four unit vectors: \mathbf{n}, \mathbf{n}' (for system A) and \mathbf{m}, \mathbf{m}' (for system B). One assumes that all the observables $\mathbf{n} \cdot \sigma$ have determined (although unknown) values ± 1 corresponding to the possible outcomes of measurements (the eigenvalues ± 1). Then the following inequality holds for a hidden variable theory (Clauser, Horne, Shimony, and Holt 1969)

$$|\langle \mathbf{n} \cdot \sigma \otimes \mathbf{m} \cdot \sigma \rangle + \langle \mathbf{n}' \cdot \sigma \otimes \mathbf{m} \cdot \sigma \rangle + \langle \mathbf{n}' \cdot \sigma \otimes \mathbf{m}' \cdot \sigma \rangle - \langle \mathbf{n} \cdot \sigma \otimes \mathbf{m}' \cdot \sigma \rangle| \leq 2 \quad (2.155)$$

where the pattern of signs is to be noted. One central idea in the proof is that one can “locally choose” between \mathbf{n} and \mathbf{n}' (i.e. two different components of the Bloch vector for system A), and that the outcome for system B is not affected by this choice (this is a “local theory of hidden variables”).

A classical, perfect correlation can always be achieved between detectors in a fixed direction, $\mathbf{n} = \mathbf{m}$, say. This is within the scope of the CHSH inequality (2.155), however. Take for example $\mathbf{n} \cdot \sigma = \sigma_3$ with eigenstates $|e\rangle, |g\rangle$ and consider the statistical mixture

$$\rho = \frac{1}{2} (|e, g\rangle\langle e, g| + |g, e\rangle\langle g, e|) \quad (2.156)$$

Then perfect anti-correlation holds $\langle \sigma_3 \otimes \sigma_3 \rangle = \text{tr}[(\sigma_3 \otimes \sigma_3) \rho] = -1$. This does not produce any correlations for the Bloch components $\sigma_{1,2}$, however. Check that one gets for the CHSH correlation

$$\begin{aligned} \text{CHSH} &= -n_3 m_3 - n'_3 m_3 - n'_3 m'_3 + n_3 m'_3 \\ &= -(n_3 + n'_3) m_3 - (n'_3 - n_3) m'_3 \end{aligned} \quad (2.157)$$

Let us look for the maximum value of this expression. The components of the unit vectors are in the range $-1 \dots +1$. For $-1 \leq n_3 < n'_3 < 0$, both parentheses are negative, and we get a maximum by choosing $m_3 = m'_3 = 1$. But then, $\text{CHSH} = -2n'_3 \leq 2$. Along similar lines, one can prove the inequality (2.155).

The power of this reasoning is that the inequality applies to any choice of state, i.e., of choice of “hidden variables” or classical correlations between the outcomes.

Quantum mechanics gives a different answer, however, sometimes. Take the “maximally entangled state”

$$|\psi\rangle = \frac{1}{\sqrt{2}} (|e, g\rangle - |g, e\rangle) \quad (2.158)$$

whose density operator $|\psi\rangle\langle\psi|$ differs from Eq.(2.156) because of the off-diagonal terms $|e, g\rangle\langle g, e|$. In this state, one has the perfect anti-correlation

$$\langle\psi|\mathbf{n} \cdot \boldsymbol{\sigma} \otimes \mathbf{n} \cdot \boldsymbol{\sigma}|\psi\rangle = -1 \quad (2.159)$$

in any choice of basis \mathbf{n} . (This is related to the “singlet” or zero total spin character of the state $|\psi\rangle$.)¹⁰ For two different orientations at A and B , one gets the result

$$\langle\psi|\mathbf{n} \cdot \boldsymbol{\sigma} \otimes \mathbf{m} \cdot \boldsymbol{\sigma}|\psi\rangle = -\mathbf{n} \cdot \mathbf{m} \quad (2.160)$$

where $\mathbf{n} \cdot \mathbf{m}$ is the standard scalar product. The CHSH correlation then becomes

$$\text{CHSH} = -(\mathbf{n} + \mathbf{n}') \cdot \mathbf{m} - (\mathbf{n}' - \mathbf{n}) \cdot \mathbf{m}' \quad (2.161)$$

which can be maximized by choosing a suitable “tetrad” of unit vectors¹¹ up to a value $2\sqrt{2}$. The range of CHSH correlations

$$2 < |\text{CHSH}| \leq 2\sqrt{2} \quad \text{Bell inequality violated} \quad (2.162)$$

is therefore called the “non-classical” domain which cannot be interpreted in terms of a classical theory (more precisely: a local hidden variable model). The number $|\text{CHSH}| - 2$ can be taken as a quantitative measure of entanglement between system A and B : it quantifies the degree of “non-classicality” of the correlations between A and B .

2.6.3 Further reading

On entanglement between quadratures or position and momentum variables in general (so-called “continuous variables”), and in Gaussian states in particular, see Eisert & Plenio (2003) and Plenio & Virmani (2007). An introduction to the EPR paradox and applications: Reid & al. (2009).

¹⁰The form of the state $|\psi\rangle$ in a different cartesian basis for the spin vectors $\boldsymbol{\sigma}$ is generated by the three operators $\sigma_i \otimes \mathbb{1} + \mathbb{1} \otimes \sigma_i$ ($i = 1, 2, 3$). However, their action on $|\psi\rangle$ gives zero: hence $|\psi\rangle$ is invariant under rotation (a “singlet state”). Hence the perfect anticorrelation for the components $\sigma_3 \otimes \sigma_3$ carries over onto any direction.

¹¹Exercise: choose \mathbf{m} and \mathbf{m}' opposite to the directions of $\mathbf{n} \pm \mathbf{n}'$, respectively. Then $\text{CHSH} = \sqrt{2 + 2\cos\theta} + \sqrt{2 - 2\cos\theta}$ with $\cos\theta = \mathbf{n} \cdot \mathbf{n}'$. This quantity varies between 2 and $2\sqrt{2}$. The maxima are obtained for $\cos\theta = 0$, hence orthogonal directions \mathbf{n} and \mathbf{n}' . The directions \mathbf{m} and \mathbf{m}' are then orthogonal as well, and one bisects that angle between \mathbf{n} and \mathbf{n}' .

2.7 Two modes, many modes

(Material not covered in SS 20.)

2.7.1 Multi-mode Hilbert space and observables

The state space of a two-mode field is the tensor product of the Fock spaces of two harmonic oscillators. In terms of number states, the basis vectors of this space can be written

$$|n_1; n_2\rangle = |n_1\rangle_{\text{mode 1}} \otimes |n_2\rangle_{\text{mode 2}}$$

where the first mode contains n_1 and the second mode n_2 photons. These states are called ‘product states’. That have expectation values of products of operators pertaining to mode 1 and 2, that factorize, e.g.,

$$\langle \hat{n}_1 \hat{n}_2 \rangle = \langle \hat{n}_1 \rangle \langle \hat{n}_2 \rangle.$$

But due to the possibility of forming superpositions, there is much more ‘space’ in the multi-mode Hilbert space. For example, it is possible that two modes ‘share’ a single photon:

$$\frac{1}{\sqrt{2}} (|0; 1\rangle + |1; 0\rangle) \quad (2.163)$$

This state is called ‘entangled’ if no change of basis for the mode expansion exists such that the state is mapped onto a product state (this may be very difficult to check in practice).¹² The state is by no means unphysical, however, since it is generated by

$$\frac{1}{\sqrt{2}} (a_1^\dagger + a_2^\dagger) |0; 0\rangle \quad (2.164)$$

where $|0; 0\rangle$ is the two-mode vacuum. Such sums of creation operators occur always in the mode expansion of the quantized field. The decay of an excited atomic state, for example, generates a continuous superposition of one-photon states where an infinite number of modes share a single photon.

Many-mode single-photon states are also generated when an atom is illuminated by a single photon: the scattering of this photon by the atom generates, as in the classical electromagnetic theory, a continuous angular distribution of modes with a nonzero amplitude for one-photon excitations.

Finally, what about the density matrix for a multi-mode field? Let us start with the simple case of two modes of the same frequency in thermal equilibrium. According to the general rule, the density matrix is a sum of projectors onto the stationary states $|n_1; n_2\rangle$ of the two-mode system, each weighted with a probability proportional to $e^{-\beta(n_1+n_2)}$. (Use

¹²It is simple to see, however, that the expectation value of $\hat{n}_1 \hat{n}_2$ does not factorize. Indeed, $\langle \hat{n}_1 \rangle = \frac{1}{2} = \langle \hat{n}_2 \rangle$ while $\langle \hat{n}_1 \hat{n}_2 \rangle = 0$ since in each component of the state (2.163), at least one mode has zero photons.

$\beta = \hbar\omega/k_B T$.) Since the energy is made additively from single-mode energies, we can factorize this density operator:

$$\begin{aligned}\hat{\rho} &= Z^{-1} \sum_{n_1, n_2} e^{-\beta(n_1+n_2)} |n_1; n_2\rangle \langle n_1; n_2| \\ &= Z \sum_{n_1} e^{-\beta n_1} |n_1\rangle \langle n_1| \otimes \sum_{n_2} e^{-\beta n_2} |n_2\rangle \langle n_2| \\ &= Z^{-1} \tilde{\rho}_1 \otimes \tilde{\rho}_2\end{aligned}\tag{2.165}$$

where the $\tilde{\rho}_{1,2}$ are un-normalized density matrices. The tensor product of the projectors is defined by coming back to the tensor product of states

$$|n_1\rangle \langle n_1| \otimes |n_2\rangle \langle n_2| = (|n_1\rangle \otimes |n_2\rangle) (\langle n_1| \otimes \langle n_2|).$$

The trace of the two-mode density matrix (2.165) also factorizes because the matrix elements of a tensor product operator are, by definition, the products of the individual matrix elements

$$\begin{aligned}\text{tr}(\hat{\rho}) &= Z^{-1} \sum_{n_1, n_2} \langle n_1; n_2 | \tilde{\rho}_1 \otimes \tilde{\rho}_2 | n_1; n_2 \rangle \\ &= Z^{-1} \sum_{n_1, n_2} \langle n_1 | \tilde{\rho}_1 | n_1 \rangle \langle n_2 | \tilde{\rho}_2 | n_2 \rangle \\ &= Z^{-1} (\text{tr} \tilde{\rho}_1) (\text{tr} \tilde{\rho}_2)\end{aligned}\tag{2.166}$$

and therefore $Z = Z_1 Z_2 = (1 - e^{-\beta})^{-2}$.

Since the density matrix of this thermal two-mode state factorizes, this state is not entangled (averages of products of single-mode operators factorize). This is no longer true, however, if we allow for an interaction between the modes. Then the energy is no longer a sum of single-mode energies, and the previous factorization does no longer work. This is by the way a general rule: interactions between quantum systems lead to entangled states. For this reason, entangled states are much more frequent in Nature than are factorized states. It is a nontrivial task, however, to decide whether a given density matrix describes an entangled state or not.

2.7.2 Digression (*Einschub*): tensor product states and operators

It is somewhat tricky to guess the right formulas for multimode field states and operators. The general rule is the following:

$$\begin{array}{ll}\text{Field operator} & \leftrightarrow \text{sum of modes} \\ \text{Field state} & \leftrightarrow \text{product of modes}\end{array}$$

For example, the electric field operator for a two-mode field is given by

$$\mathbf{E}(\mathbf{x}, t) = E_1 \boldsymbol{\varepsilon}_1 a_1(t) e^{i\mathbf{k}_1 \cdot \mathbf{x}} + E_2 \boldsymbol{\varepsilon}_2 a_2(t) e^{i\mathbf{k}_2 \cdot \mathbf{x}} + \text{h.c.}$$

while a typical state is for example the product state $|n_1; n_2\rangle = |n_1\rangle \otimes |n_2\rangle$. The general rule gets complicated (1) when we allow for superpositions (sums) of product states and (2) when we consider measurements that involve products of different mode operators.

In calculations, one often needs products of operators, like $\mathbf{E}^2(\mathbf{x}, t)$. These are computed in the usual way, one has just to take care that operators sometimes do not commute. But this is only relevant for operators acting on the same mode, $[a_1, a_1^\dagger] = 1$, while for different modes

$$[a_1, a_2^\dagger] = 0$$

because they correspond to independent degrees of freedom.

Operator averages in product states. Let us consider the average electric field for the two-mode case written above. Using the mode expansion, we find terms like $\langle a_i(t) \rangle$ ($i = 1, 2$) and their adjoints. Now the operator $a_1|\psi\rangle$ is evaluated by letting a_1 act on the first factor of a product state:

$$a_1|n_1; n_2\rangle = (a_1|n_1\rangle) \otimes |n_2\rangle$$

If $|\psi\rangle$ is a sum of product states (entangled state), then this procedure is done for every term in this sum. Sometimes this is formalized by writing the operator as $a_1 \otimes \mathbb{1}$, thus indicating that for the second mode nothing happens. The action of such operator tensor products is apparently defined as

$$A_1 \otimes B_2|n_1; n_2\rangle = A_1|n_1\rangle \otimes B_2|n_2\rangle \quad (2.167)$$

by letting each operator factor act on the respective state factor. This notation allows to avoid the subscripts 1 and 2 as the relevant mode is indicated by the position in the operator product.

Similarly, the scalar product of tensor products of states is defined by

$$\langle n_1; n_2 | m_1; m_2 \rangle = \langle n_1 | \otimes \langle n_2 | m_1 \rangle \otimes | m_2 \rangle = \langle n_1 | m_1 \rangle \langle n_2 | m_2 \rangle$$

by taking the scalar product of the corresponding factors.

The average of the electric field for a product of number states is thus zero, as for a single-mode field, because $\langle n | a | n \rangle = 0$, and this is true for both modes. What about a product state of two coherent states, $|\psi\rangle = |\alpha; \beta\rangle$? It is simple to see that we get the classical result (we assume that both modes have the same frequency ω)

$$\langle \mathbf{E}(\mathbf{x}, t) \rangle = E_1 \boldsymbol{\varepsilon}_1 \alpha e^{-i\omega t + i\mathbf{k}_1 \cdot \mathbf{x}} + E_2 \boldsymbol{\varepsilon}_2 \beta e^{-i\omega t + i\mathbf{k}_2 \cdot \mathbf{x}} + \text{c.c.} \quad (2.168)$$

(Note that ‘c.c.’ and not ‘h.c.’ occurs.) As a general rule, classical fields can be described by tensor products of coherent states.

Last example where we go quantum: a superposition of coherent product states,

$$|\psi\rangle = c|\alpha; \beta\rangle + d|\beta; \alpha\rangle$$

with some complex amplitudes c, d . Then we find

$$\langle a_1 \rangle = |c|^2 \alpha + |d|^2 \beta$$

if $\langle \alpha | \beta \rangle = 0$. (This is actually never exactly the case, but can be achieved to a very good precision if $|\alpha - \beta| \gg 1$.) This result is an average over the two possible coherent amplitude, weighted with the corresponding probabilities. The average field thus becomes:

$$\langle \mathbf{E}(\mathbf{x}, t) \rangle = E_1 \boldsymbol{\varepsilon}_1 (|c|^2 \alpha + |d|^2 \beta) e^{-i\omega t + i\mathbf{k}_1 \cdot \mathbf{x}} + E_2 \boldsymbol{\varepsilon}_2 (|c|^2 \beta + |d|^2 \alpha) e^{-i\omega t + i\mathbf{k}_2 \cdot \mathbf{x}} + \text{c.c.}$$

Question: this result does not allow to distinguish this state from an ‘incoherent mixture’ of coherent product states like in (2.168), each state occurring with a probability $|c|^2, |d|^2$. This mixture would be described by the density operator

$$\hat{\rho}_{\text{mix}} = |c|^2 |\alpha; \beta\rangle \langle \alpha; \beta| + |d|^2 |\beta; \alpha\rangle \langle \beta; \alpha|$$

and gives the same average electric field (exercise). If the coherent amplitudes α, β are closer together, then due to the nonzero overlap $\langle \alpha | \beta \rangle$, one can distinguish superposition and mixture (exercise). Are there observables that can make the difference in the case $\langle \alpha | \beta \rangle = 0$?

Average of single-mode operator. Let us calculate as another example the average photon number in mode 1 for a two-mode field in the entangled state (2.163). The relevant photon number operator is given by $a_1^\dagger a_1$ or, to be more precise, $a_1^\dagger a_1 \otimes \mathbb{1}$. Its action on the entangled state is worked out using linearity and the operator product rule (2.167)

$$\begin{aligned} & \frac{1}{\sqrt{2}} a_1^\dagger a_1 \otimes \mathbb{1} (|0; 1\rangle + |1; 0\rangle) \\ &= \frac{1}{\sqrt{2}} (a_1^\dagger a_1 |0\rangle \otimes |1\rangle + a_1^\dagger a_1 |1\rangle \otimes |0\rangle) \\ &= \frac{1}{\sqrt{2}} |1\rangle \otimes |0\rangle = \frac{1}{\sqrt{2}} |1; 0\rangle \end{aligned}$$

Taking the scalar product with the original state, we find

$$\langle \hat{n}_1 \rangle = \frac{1}{2} (\langle 0; 1| + \langle 1; 0|) |1; 0\rangle = \frac{1}{2}.$$

Once you have done this calculation, you can use the shorter rule: all we need are the probabilities of having $n_1 = 0, 1, \dots$ photons in mode 1. For this, collect all product states in the state with the same number of photons n_1 and compute the squared norm of these states. From the probabilities for n_1 photons, you get the average photon number.

Product operators. As a second example, let us compute the average value of the product $a_i^\dagger a_j$ ($i, j = 1, 2$) in a thermal two-mode state. This object occurs when you

measure the two-mode field with a photodetector (see paragraph ?? below). The tensor product notation is more cumbersome here and gives

$$a_1^\dagger a_1 \otimes \mathbb{1} \quad \text{or} \quad \mathbb{1} \otimes a_2^\dagger a_2 \quad \text{or} \quad a_1^\dagger \otimes a_2 \quad \text{or} \quad a_1 \otimes a_2^\dagger.$$

The density matrix is a tensor product of thermal single-mode density matrices. We shall see that the result is:

$$\langle a_i^\dagger a_j \rangle_T = \delta_{ij} \bar{n}(T) \quad (2.169)$$

where $\bar{n}(T)$ is the average photon number in a single mode. How does this come about?

When $i = j$, we are left with the calculation of the average photon number for a single mode:

$$\langle a_i^\dagger a_i \rangle = \sum_{n_1, n_2} \langle n_1; n_2 | a_i^\dagger a_i \hat{\rho}_1 \otimes \hat{\rho}_2 | n_1; n_2 \rangle$$

The action of the product density operators factorizes:

$$\hat{\rho}_1 \otimes \hat{\rho}_2 | n_1; n_2 \rangle = \hat{\rho}_1 | n_1 \rangle \otimes \hat{\rho}_2 | n_2 \rangle$$

Each single-mode density operator, acting on a number state, gives the corresponding occupation probability:

$$\hat{\rho}_1 | n_1 \rangle = \sum_{m_1} p_{m_1}(T) | m_1 \rangle \langle m_1 | n_1 \rangle = p_{n_1}(T) | n_1 \rangle,$$

so that we have, using the result for the photon number of one mode

$$\begin{aligned} \langle a_i^\dagger a_i \rangle &= \sum_{n_1, n_2} p_{n_1}(T) p_{n_2}(T) \langle n_1; n_2 | a_i^\dagger a_i | n_1; n_2 \rangle \\ &= \sum_{n_1, n_2} p_{n_1}(T) p_{n_2}(T) n_i \\ &= \sum_{n_i} p_{n_i}(T) n_i \sum_{n_j} p_{n_j}(T) \end{aligned}$$

In the last step, we have noted that the double sum can be factorized ($j \neq i$ is the other index). The second sum gives unity because the probabilities are normalized, the first sum gives the average photon number $\bar{n}(T)$ at temperature T and does no longer depend on the mode label (this is because we assumed equal frequencies for both modes). This completes the proof in the case $i = j$.

A similar calculation shows that the average of $a_1^\dagger a_2$ vanishes: indeed, we have

$$\langle n_1; n_2 | a_1^\dagger a_2 | n_1; n_2 \rangle = \langle n_1 | a_1^\dagger | n_1 \rangle \langle n_2 | a_2 | n_2 \rangle = 0.$$

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