Chapter 3

Quantum theory of the laser

Vorläufige Fassung vom 20 Mai 2020. Soll ab und zu aktualisiert werden.

3.1 Introduction

(Material covered in part in SS 2020.)

What are the typical components of a laser[1]? Without going into details, we can identify two of them:

- some matter that amplifies light ("active medium");
- some device that traps the light around the space filled with the medium ("cavity")

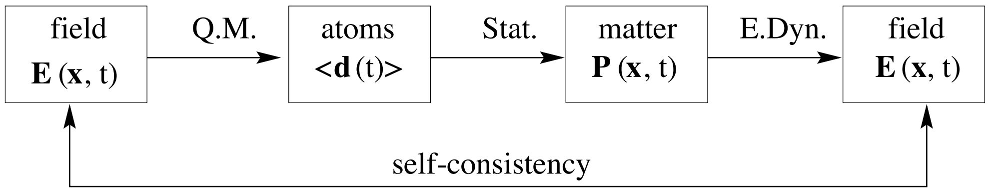
In order to obtain an amplifying medium, one has to "pump" energy into it. The gain medium is thus a converter between the pump energy and the light emission. Quite often, the conversion efficiency is low, with values in the range 10–50% being considered "large".

The feedback mechanism is needed because the light would otherwise escape from the medium. An optical cavity like a Fabry-Pérot resonator (two mirrors) does this job because the light can travel back and forth between the mirrors a large number of times.

General ideas

Sargent and Scully (1972) propose the following diagram that relates the different theories needed to describe a laser.

[1] Acronym of "Light Amplifier by Stimulated Emission of Radiation"



The electromagnetic field drives microscopic dipoles in the laser medium. This was the topic of the previous term. A statistical description gives, implicit in the density matrix approach we followed, links the dipoles to the macroscopic polarization of the medium [see Eq. (3.14) below]. The polarization enters the Maxwell equations for the electromagnetic field as a source and generates the field. In the end, a self-consistent description is required: the fields at the left and right end should coincide. The condition of self-consistency allows to derive the following important quantities:

- the laser threshold,
- the laser intensity in steady state,
- the laser frequency.

This can be achieve even when one treats the field treated classically. For example, a "classical" or "coherent" field appeared via the Rabi frequency in the Bloch equations of the previous term. This approach is called "semi-classical laser theory". Note that nowhere in this approach does the word "photon" appear (if one is serious).

When a quantum-mechanical description is adopted, the photon finally comes into play and one may also derive

- the photon number probabability distribution ("photon statistics"),
- the intensity fluctuations and correlations of laser light,
- the phase fluctuations (related to the laser linewidth).

We shall illustrate this quantum theory by a calculation of the photon statistics and the laser linewidth.

We focus in these elementary considerations on a homogeneously distributed medium in the cavity made up from identical two-level systems ("homogeneous broadening"). Please refer to the experimental physics lectures for the discussion of "inhomogeneous" frequency broadening due to, for example, the atomic motion and other features.

3.2 Cavity field

Reduction of the full electric field operator to a single mode:

$$\mathbf{E}(\mathbf{r}, t) = \mathcal{E}_{1\mathrm{ph}} \hat{a}(t) \mathbf{f}(\mathbf{r}) + \mathrm{h.c.} \tag{3.1}$$

where the 'mode function' $\mathbf{f}(\mathbf{r})$ describes the spatial pattern of the cavity mode. It is the result of classical electrodynamics. A simplified form could be

$$\mathbf{f}(\mathbf{r}) \approx \mathbf{e}\mathcal{N}\frac{\sin(kz)}{\sqrt{Lw^2}}\mathrm{e}^{-(x^2+y^2)/w^2} \tag{3.2}$$

where $\mathbf{e}$ is a polarization vector and $\mathcal{N}$ is a numerical factor determined by the normalization integral

$$1 \stackrel{!}{=} \int\limits_V \mathrm{d}V\, |\mathbf{f}(\mathbf{r})|^2 \tag{3.3}$$

The volume V of the cavity has length L in the z-direction. Mode functions like Eq.(3.2) appear naturally in the so-called paraxial approximation where one assumes that the mode is concentrated around the cavity axis (the z-axis). The 'waist' w gives the size transverse to the axis (in the xy-plane), in the paraxial approximation, we have $kw \gg 1$. The mode function $\mathbf{f}(\mathbf{r})$ solves a wave equation for a fixed frequency (vector Helmholtz equation); we call this the 'cavity frequency' ω_c in the following.

The electric field $\mathbf{E}(\mathbf{r}, t)$ becomes an operator through the time-dependent amplitude $\hat{a}(t)$ and its hermitean conjugate $\hat{a}^\dagger(t)$. These operators annihilate ($\hat{a}(t)$) and create ($\hat{a}^\dagger(t)$) photons in the cavity – they act as ladder operators in the same way as for the harmonic oscillator of elementary quantum mechanics. Indeed, the energy spectrum of the cavity is given by the Hamiltonian operator

$$\hat{H}_c = \hbar\omega_c\hat{a}^\dagger(t)\hat{a}(t) \tag{3.4}$$

The basic relation between the creation and annihilation operators is the commutator

$$\left[\hat{a}(t),\, \hat{a}^\dagger(t)\right] = 1 \tag{3.5}$$

where 1 denotes the unit operator. This relation holds only if both operators are evaluated at the same time. Remember that from this, one gets that the eigenvalues of $\hat{a}^\dagger(t)\hat{a}(t)$ are $0, 1, 2\ldots$: this operator is called the

$$\text{number operator } \hat{n} = \hat{a}^\dagger(t)\hat{a}(t) \tag{3.6}$$

and that the energy eigenvalues are

$$E_n = \hbar\omega_c n \tag{3.7}$$

Up to the choice of the ground state energy, the excited states of the cavity can therefore be understood as carrying a number $n = 1, 2\ldots$ of energy quanta $\hbar\omega_c$, one per photon.

Free evolution of the cavity field. In the Schrödinger picture, in the number state basis $|n\rangle$

$$|\psi(0)\rangle = \sum_{n=0}^{\infty} c_n|n\rangle \quad \to \quad |\psi(t)\rangle = \sum_{n=0}^{\infty} c_n\mathrm{e}^{-\mathrm{i}n\omega_c t}|n\rangle \tag{3.8}$$

For the density operator $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$

$$\mathrm{i}\hbar\partial_t\rho = [H_c,\, \rho] \quad \to \quad \rho_{nm}(t) = \rho_{nm}(0)\,\mathrm{e}^{-\mathrm{i}(n-m)\omega_c t} \tag{3.9}$$

where the matrix elements are denoted

$$\rho_{nm} = \langle n|\rho|m\rangle \tag{3.10}$$

Observe that the *populations* ρ_{nn} remain constant, while the *coherences* ρ_{nm} with $n \neq m$ oscillate at multiples of the cavity frequency.

In particular, for the expectation value of the cavity field, we need to compute (in the Schrödinger picture)

$$\begin{aligned}
\langle \hat{a}(t)\rangle &= \langle \psi(t)|a|\psi(t)\rangle = \mathrm{tr}[a\rho(t)] \\
&= \sum_n \langle n|a\rho(t)|n\rangle = \sum_{n,m} \langle n|a|m\rangle\langle m|\rho(t)|n\rangle \\
&= \sum_{n,m} \delta_{n-1,m}\sqrt{m}\rho_{mn}(t) = \sum_m \sqrt{m}\rho_{m,m-1}(t)
\end{aligned} \tag{3.11}$$

See how the average field involves this sum over coherences. Inserting the free cavity evolution, we find

$$\langle \hat{a}(t)\rangle = \sum_m \sqrt{m}\rho_{m,m-1}(0)\mathrm{e}^{-\mathrm{i}\omega_c t} \tag{3.12}$$

where the exponential contains just once the cavity frequency.

Exercise. In the Heisenberg picture, $\hat{a}(t) = \hat{a}(0)\,\mathrm{e}^{-\mathrm{i}\omega_c t}$ which gives the same result, from a slightly shorter calculation. Similarly, $\hat{n}(t) = \hat{n}$ for a free cavity.

Note. We now have to add to the description the active medium and the loss from the cavity (to get at least some laser beam out of the device).

3.3 Active medium

The "active medium" cavity inside the cavity that provides the polarization consists, in many cases, of a large number of atoms or molecules. These atoms are prepared in the excited state by some process that feeds energy into them ("pumping mechanism"), and then wait to release their energy in the form of photons into the cavity field. A two-level approximation for the atoms is a simple way to account for the sharp, nearly monochromatic emission spectrum of the laser. We could have used as well a harmonic oscillator[2], however, this does not reproduce some basic features of the laser like gain saturation. The quantum theory for the atom-light interaction gives us an expression for the "microscopic", average electric dipole $\langle \mathbf{d}(t)\rangle$. The polarization field is then simply the number density of these dipoles

$$\mathbf{P}(\mathbf{x},t) = N(\mathbf{x})\langle \mathbf{d}(t)\rangle. \tag{3.13}$$

In general, the density $N(\mathbf{x})$ is position dependent. In fact, also the induced dipole is because it involves the light field at the position $\mathbf{x}$.

[2]This is a good model for antennas emitting at radio frequencies.

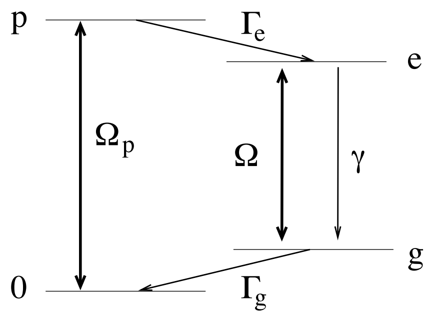


Figure 3.1: Four-level model to describe incoherent population pumping of the upper state e of the lasing transition e $\leftrightarrow$ g.

The complex, slowly varying polarization field can be connected to the coherences of the density matrix in the rotating frame:

$$\begin{aligned} \mathbf{P}(\mathbf{x}, t) &= N(\mathbf{x})\left[\langle \mathbf{d}^{(+)}(t)\rangle + \text{c.c.}\right] \\ &= N(\mathbf{x})\left[\mathbf{d}_{\text{ge}}\, \rho_{\text{eg}}(t)\text{e}^{-\text{i}\omega_L t} + \text{c.c.}\right] \end{aligned} \tag{3.14}$$

where $\mathbf{d}^{(+)}(t)$ is the positive frequency part of the atomic dipole operator, $\mathbf{d}_{\text{ge}}$ is the fixed vector of dipole matrix elements and $\rho_{\text{eg}}(t)$ is the off-diagonal element ("optical coherence") of the atomic density matrix in the frame rotating at ω_L. A formula like (3.14) assumes that all dipoles in the medium are driven by a similar field and do not interact with each other. This is a first starting point and leaves place for more elaborate theories, of course. We assume in particular that the resonance frequency is the same for all microscopic dipoles. This is not true for atoms in a (thermal) gas where the Doppler effect leads to a distribution of the resonance frequencies ("inhomogeneous broadening").

Polarization and saturation

We now have to find a way to determine the dipole moment of the two-level atoms. It will turn out that its imaginary part, essential for light amplification, depends on the atomic inversion (population difference between upper and lower state). To this end, we use the optical Bloch equations for the atomic density matrix, with some modifications by adding additional energy levels. This model also provides a better understanding of the "pumping" mechanism, beyond some phenomenological rate equations. The modified two-level system is for example a four-level atom with fast relaxation in the two upper and two lower states. A simple model with four states as shown in figure 3.1 is outlined in the exercises. In this limit, the optical Bloch equations can be simplified, and one gets a justification for the often-used rate equations.

The next task is to compute the optical coherence $\rho_{\text{eg}}(t)$ from the optical Bloch equations. Let us write down these equations for the two levels e and g involved in the laser transition. The rate equations for the populations $\rho_{\rm ee}$ and $\rho_{\rm gg}$ involve a pumping rate $\lambda_{\rm e} = \Gamma_{\rm e}\rho_{\rm pp}$ into the upper state (via rapid decay from the pumped state p), the spontaneous decay rate γ and a decay rate $\gamma_{\rm g}$ for the lower state. Including the Rabi frequency $\Omega = -(2/\hbar)\mathbf{d}\cdot\mathbf{E}$ for the laser field, as you have learned in the previous semester, this gives

$$\dot\rho_{\rm ee} = \lambda_{\rm e} - \gamma\rho_{\rm ee} + {\rm i}\frac{\Omega}{2}\left(\rho_{\rm eg} - \rho_{\rm ge}\right), \tag{3.15}$$

$$\dot\rho_{\rm gg} = \gamma\rho_{\rm ee} - \gamma_{\rm g}\rho_{\rm gg} - {\rm i}\frac{\Omega}{2}\left(\rho_{\rm eg} - \rho_{\rm ge}\right). \tag{3.16}$$

Notice that the first equation gives an increase of the excited state population when the coherence $\rho_{\rm eg}$ has a positive imaginary part (recall that Ω is actually negative. . .). This is in agreement with the damping of the field energy in the wave equation derived before.

The last Bloch equation is for the coherence itself. The population decay rates γ and $\gamma_{\rm g}$ lead to a decoherence rate $\Gamma = \frac{1}{2}(\gamma + \gamma_{\rm g})$ as you have seen in the derivation of the Bloch equations. In the frame rotating at the laser frequency ω_L, the laser field detuning is $\Delta = \omega_L - \omega_{\rm eg}$, and we get

$$\dot\rho_{\rm eg} = {\rm i}\Delta\rho_{\rm eg} - \Gamma\rho_{\rm eg} + {\rm i}\frac{\Omega}{2}\left(\rho_{\rm ee} - \rho_{\rm gg}\right). \tag{3.17}$$

From this equation we learn that the optical dipole is created by the population difference (inversion) $\rho_{\rm ee} - \rho_{\rm gg}$. As discussed in the exercises, this equation can be solved approximately in the limit that the decay rate Γ is the largest time constant around (this solution also corresponds to the stationary state):

$$\rho_{\rm eg} = -\frac{\Omega/2}{\Delta + {\rm i}\Gamma}\left(\rho_{\rm ee} - \rho_{\rm gg}\right). \tag{3.18}$$

In particular, the imaginary part of the optical coherence is (we assume as usual a real Rabi frequency)

$$\operatorname{Im}\rho_{\rm eg} = \frac{\Gamma\Omega/2}{\Delta^2 + \Gamma^2}\left(\rho_{\rm ee} - \rho_{\rm gg}\right).$$

Note that this expression is negative when the two-level system is inverted (upper level population $\rho_{\rm ee} > \rho_{\rm g}$), using again that actually $\Omega < 0$. This means that the medium amplifies the light via stimulated emission.

Solving also the other Bloch equations in the stationary state, we can compute the inversion

$$\rho_{\rm ee} - \rho_{\rm gg} = \lambda_{\rm e}\left(\frac{1}{\gamma} - \frac{1}{\gamma_{\rm g}}\right)\frac{\Delta^2 + \Gamma^2}{\Delta^2 + \Gamma^2 + (\Gamma/\gamma)\Omega^2/2}. \tag{3.19}$$

The system is inverted when the lifetime $1/\gamma$ of the upper state exceeds the lifetime $1/\gamma_{\rm g}$ of the lower state, which is perfectly reasonable.

The end result of the calculation is the following expression for the polarization field. We quote only the amplitude of the positive frequency component and assume that dipole moment and electric field are collinear:

$$\mathbf{P}(\mathbf{x}) = N(\mathbf{x})(D^2/\hbar)\mathbf{e}E(\mathbf{x})\frac{\lambda_{\rm e}\left(1/\gamma - 1/\gamma_{\rm g}\right)(\Delta - {\rm i}\Gamma)}{\Delta^2 + \Gamma^2 + 2(\Gamma D^2/\gamma\hbar^2)|E(\mathbf{x})|^2} \tag{3.20}$$

or

$$\boxed{\mathbf{P}(\mathbf{x}) = \frac{\varepsilon_0\chi\mathbf{E}(\mathbf{x})}{1 + B|E(\mathbf{x})|^2}.} \tag{3.21}$$

In the last line, we have introduced the (linear) susceptibility χ of the laser medium and a coefficient B that takes into account the nonlinear response.

The most important result is that the imaginary part of the polarization is negative (*amplification* of the field) when the two-level system is inverted. The four-level scheme shown in fig. 3.1 is just one possibility to achieve inversion by a suitable pumping scheme. Refer to the experimental physics lectures for other, perhaps more efficient, pumping mechanisms.

The coefficient B in (3.21) describes the *saturation* of the medium: for very large laser intensity $|E|^2$, the induced polarization decreases proportional to $1/|E|$ instead of increasing. The physics behind saturation is characteristic for the two-level system: when the laser field gets extremely strong, the inversion vanishes (see Eq. (3.19)). We have already seen this behaviour when we considered Rabi oscillations with weak damping: the two-level system gets always re-excited by the laser and is finally with equal probability in the upper and lower states. For a harmonic oscillator, there is no saturation since arbitrarily high lying states can be populated.

3.4 Semiclassical theory of the laser field

We start with a reminder of the electrodynamics in a material with a given polarization. Let us recall that the polarization field $\mathbf{P}$ enters the following Maxwell equation:

$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{j} + \frac{\partial}{\partial t}\left(\varepsilon_0 \mathbf{E} + \mathbf{P}\right)$$

where it gives the "bound" part of the current density. We put in the following the "free" current density $\mathbf{j} = \mathbf{0}$ because we assume that the active material is globally neutral and the light only induces dipoles in it. Combining with the Faraday induction equation, $\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$, one gets the wave equation for the electric field where the polarization enters as a source term:

$$\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = -\mu_0 \frac{\partial^2}{\partial t^2}\mathbf{P}. \tag{3.22}$$

We now make the approximation that in the cavity, a single mode is sufficient to capture the field dynamics. You have seen that one can then write for the field operator

$$\mathbf{E}(\mathbf{x}, t) = \mathcal{E}_{\rm 1ph}\left[\mathbf{f}(\mathbf{x})\, a(t) + \mathbf{f}^*(\mathbf{x})\, a^\dagger(t)\right] \tag{3.23}$$

where $\mathcal{E}_{\rm 1ph} = (\hbar\omega_L/2\varepsilon_0)^{1/2}$ is the "one-photon field amplitude" (up to a factor $1/\sqrt{V}$ where V is the 'mode volume' that appears in $\mathbf{f}$), $a(t)$ is the annihilation operator for the mode (time-dependent in the Heisenberg picture), and $\mathbf{f}(\mathbf{x})$ is the spatial mode function. It solves the homogeneous equation

$$\nabla \times \nabla \times \mathbf{f} - \frac{\omega_c^2}{c^2}\mathbf{f} = \mathbf{0} \tag{3.24}$$

as you remember from the field quantization procedure. Here ω_c is one of the (empty) cavity resonance frequencies. The mode function is normalized such that the integral of its square over the cavity volume V gives 1, Eq.(3.3). There are lasers where propagating modes are a suitable description. In the photonics lectures, other cavity modes, including their transverse behaviour (perpendicular to the cavity axis) are introduced. For the semiclassical theory we develop here first, the product $\mathcal{E}_{1\mathrm{ph}} a(t) = E(t)$ gives the (positive frequency) field amplitude. Its absolute square corresponds to the intensity, with $a^*(t)a(t)$ giving the "photon number" (although this is not required to be an integer in the semiclassical theory).

We now project the wave equation (3.22) onto the field mode $\mathbf{f}(\mathbf{x})$. The term involving $\mathbf{f}^*(\mathbf{x})a^\dagger(t)$ does not contribute when a propagating mode is used. We also switch to $E(t) \mapsto E(t)\,\mathrm{e}^{-\mathrm{i}\omega_L t}$ and make the approximation that $E(t)$ is a slowly varying envelope. Same notation for the polarization field, $\mathbf{P}(\mathbf{x},t) = \mathbf{P}(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\omega_L t} + \mathrm{c.c.}$ This means that the time derivative of $\mathbf{P}(\mathbf{x})$ is much smaller than $\omega_L \mathbf{P}(\mathbf{x})$. (With this approximation and standing wave modes, the term $\mathbf{f}^*(\mathbf{x})a^\dagger(t)$ drops out now.) We get

$$\dot{E} = -\mathrm{i}(\omega_L - \omega_c)E - \frac{\kappa}{2}E + \mathrm{i}\frac{\omega_L}{2\varepsilon_0}\int\limits_V \mathrm{d}^3x\,\mathbf{f}^*(\mathbf{x})\cdot\mathbf{P}(\mathbf{x}), \tag{3.25}$$

where $\omega_L - \omega_c$ is the frequency detuning with respect to the cavity resonance and V the cavity volume. We have introduced the phenomenological decay rate κ for the energy of the cavity field. The quality factor of the cavity (often known experimentally) is given by $Q = \omega_c/\kappa$. Notice that the spatial integral in Eq.(3.25) is the overlap of the polarization field with the cavity mode. It is easy to see from this equation that the *real* part of the polarization $\mathbf{P}$ determines a frequency shift of the laser (with respect to the cavity frequency), and that its *imaginary* part changes the energy $\propto |E|^2$ of the field. In particular, if $\mathrm{Im}\,\mathbf{P}$ is negative, the field energy increases (emission). We thus anticipate to find the absorption and emission of the medium in the imaginary part of the polarization.

We now want an equation for the intensity $I(t) = |E(t)|^2$ (varies also slowly in time) in the mode $\mathbf{f}(\mathbf{x})$. To this end, we work out the spatial overlap in Eq. (3.25) with a standing wave mode $\mathbf{f}(\mathbf{x}) \sim \mathbf{e}\,L^{-1/2}\sin kz$:

$$\dot{E} = -\mathrm{i}(\omega_L - \omega_c)E - \frac{\kappa}{2}E + \mathrm{i}\frac{\omega_L\chi}{2}E(t)\int\frac{\mathrm{d}z}{L}\frac{2\sin^2(kz)}{1 + 2B|E(t)|^2\sin^2(kz)}$$

Here, L is the cavity length. The difficulty is the sine function in the denominator. In the exercises, you are asked to compute this integral analytically. Here, we adopt an approximate treatment that is also often used in the literature and assume that the saturation is weak. The denominator can then be expanded, and to first order in B, we get

$$\int\frac{\mathrm{d}z}{L}2\sin^2(kz)\left[1 - 2B|E(t)|^2\sin^2(kz)\right] = 1 - \frac{3B}{2}|E(t)|^2$$

As an exercise, you can estimate the dimensionless quantity $B|E(t)|^2$ for typical parameter values. This result is often "resummed" to make the saturation effect more clear:

$$1 - \frac{3B}{2}|E(t)|^2 \approx \frac{1}{1 + \frac{3B}{2}|E(t)|^2}$$

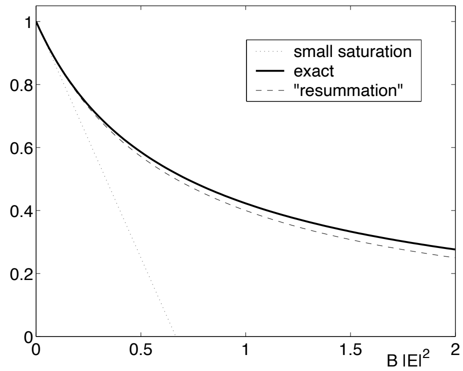


Figure 3.2: Different approximations for the gain saturation factor, integrated over the cavity mode.

This procedure may seem strange, but reproduces quite well the exact result, as shown in figure 3.2.

The equation of motion for the intensity is now given by (for small gain saturation; check that B is real)

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 2\,\mathrm{Re}\left(E^* \frac{\mathrm{d}E}{\mathrm{d}t}\right) = -\kappa I(t) - \omega_L (\mathrm{Im}\,\chi) I(t) \left[1 - \frac{3BI(t)}{2}\right] \tag{3.26}$$

This form suggests the introduction of an amplification rate ("gain") $G = -\omega_L(\mathrm{Im}\,\chi)$. Using also the conventional notation $\beta = 3GB/2$ (a rate per intensity) for the saturation coefficient, we obtain

$$\frac{\mathrm{d}I}{\mathrm{d}t} = (G - \kappa)I - \beta I^2 \tag{3.27}$$

as the fundamental equation of motion for the laser intensity in the semiclassical theory.

Exercise. Laser threshold and dependence of steady-state intensity I_{ss} on the gain. Dotted: result of quantum theory: the threshold becomes smooth.

3.5 Scully-Lamb master equation

(Material covered in SS 2020.)

In this section, we outline a theory of the laser that starts from a quantum description of the cavity field. We still use for simplicity the single mode approximation — the basic observables are hence the annihilation and creation operators $a, a^\dagger$ for the field mode.

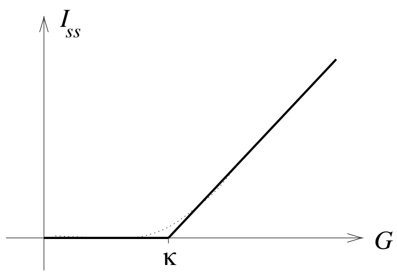


Figure 3.3: Laser intensity vs. gain G. The lasing threshold appears where G and the cavity loss rate κ coincide.

The laser is an *open quantum system* because energy is continuously fed into and removed from the cavity mode. We therefore have to use a density matrix description, as we did in the first part for a two-level atom. What are the "reservoirs" that the field mode is coupled to? First of all, the mode continuum outside the cavity: part of the cavity losses show up here (and permit to observe the laser dynamics). But in general, losses also occur in the material that makes up the cavity: mirrors and optical elements. We do not develop in this semester's course a detailed quantum theory of lossy optical elements (see earlier versions). Finally, the laser medium is also a reservoir of energy that may flow into the field mode — or not when the medium spontaneously emits photons into other modes.

In this section, we recall the master equation description for linear cavity loss and motivate the corresponding model for the gain medium. We shall derive a rate equation for the probabilities of finding n photons in the laser mode whose stationary solution gives the photon statistics. Finally, a sketch is given of the Schawlow-Townes limit for the laser linewidth.

3.5.1 Density operator and master equation

The density operator for the cavity field, $\rho(t)$, acts on the Hilbert space for the harmonic oscillator associated with the field mode. Taking the trace, we find the quantum expectation values of the quantities of interest. The average electric field, for example, is given by

$$\langle \mathbf{E}(\mathbf{x}, t) \rangle = \mathbf{f}(\mathbf{x}) \mathcal{E}_{1\text{ph}} \langle a(t) \rangle + \text{c.c.} = \mathbf{f}(\mathbf{x}) \mathcal{E}_{1\text{ph}} \, \text{tr} \left[a \, \rho(t) \right] + \text{c.c.}$$

The trace can be performed in any basis, using photon number states or coherent states, for example. In the absence of any interaction, the Heisenberg operator a evolves freely at the frequency $\omega_{\rm c}$ of the cavity. (We suppose for simplicity that this coincides with the laser frequency.)

We shall use a formalism called 'Lindblad master equation' to describe damping and gain in the laser. This is a technique that generalizes the time-dependent Schrödinger equation[3] to include the coupling to the 'rest of the world': the laser (or any other 'system') can lose energy and information, but in many cases, we do not allow for the system to 'disappear'. This means that quantum-mechanical probability is conserved. And of course, we want to keep the general formalism with positive probabilities.

A Lindblad master equation involves so-called '(Lindblad) jump operators' L that have an intuitive interpretation in terms of changes in the state of the laser. If there are several (L_k) of them, the master equation is

$$\partial_t \rho = \frac{1}{\mathrm{i}\hbar}[H, \rho] + \sum_k \left[L_k \rho L_k^\dagger - \{ L_k^\dagger L_k,\, \rho \} \right] \,, \tag{3.28}$$

3.5.2 Cavity damping

The jump operator we need for cavity damping is given by $L = \sqrt{\kappa}\, a$, leading to

$$\left.\frac{\mathrm{d}\rho}{\mathrm{d}t}\right|_{\rm damp} = \kappa\, a\rho a^\dagger - \frac{\kappa}{2} \left\{ a^\dagger a,\, \rho \right\} . \tag{3.29}$$

It is easy to check that the rate κ has the same meaning as in the semiclassical theory: it gives the (exponential) decay of the field's photon number if no other dynamics is present.

As an exercise, you may want to derive the rate equations for the diagonal elements $p_n(t) = \langle n|\rho(t)|n\rangle$ of the density matrix (they represent the 'photon statistics'). The master equation (3.29) gives a transition rate between the photon number states $|n\rangle$ and $|n-1\rangle$ that is given by $n\kappa$, proportional to the number of photons that are presently in the cavity mode. One is tempted to interpret this as "each photon decides independently to

[3] Actually, we deal here with so-called mixed states or density operators ρ. They follow the von Neumann equation of motion when the system is closed, $\mathrm{i}\hbar\partial_t \rho = [H, \rho]$ where H is the Hamiltonian.

leave the cavity." The final state is the vacuum state with zero photons — this is related to the implicit assumption that the reservoir is at zero temperature. It is a reasonable approximation at optical frequencies and room temperature.

3.5.3 Gain

In the previous semester, we used a model with a driven field mode where a "pump" generates a coherent state. We cannot use this model any longer because the laser medium does not provide, a priori, a fixed phase reference for the field it generates. At least the spontaneous emission of the pumped two-level atoms is "incoherent" (no fixed phase).

A suitable model for cavity gain in the linear regime is given by the jump operator $L_2 = \sqrt{G}\, a^\dagger$ and the master equation

$$\left.\frac{\mathrm{d}\rho}{\mathrm{d}t}\right|_{\mathrm{gain}} = G\, a^\dagger \rho a - \frac{G}{2} \left\{ a a^\dagger,\, \rho \right\}, \tag{3.30}$$

where G is the gain rate coefficient and, up to the exchange of a and $a^\dagger$, no sign changes occur.

What about gain saturation? It is included in this theory if we allow G to depend on the instantaneous intensity of the cavity mode. The gain thus depends on the photon number, $G = G(n)$. By analogy to the semiclassical gain, one can use the model

$$G(a^\dagger a) = \frac{G_0}{1 + B a^\dagger a} \tag{3.31}$$

where $1/B$ plays the role of a saturation photon number. This actually leads to complications in the construction of suitable Lindblad operators. (How to take the square root here?) There are examples in textbooks that work with non-Lindblad forms and that need to be corrected afterwards to avoid unphysical results like negative probabilities. A discussion of this problem for a simple model ('micro maser') is sketched in Sec.3.7.

To summarize, combining cavity losses and gain, we have the following (approximate) master equation

$$\begin{aligned} \frac{\mathrm{d}\rho}{\mathrm{d}t} &= -\mathrm{i}\omega_{\mathrm{c}} \left[a^\dagger a,\, \rho \right] + \kappa\, a \rho a^\dagger - \frac{\kappa}{2} \left\{ a^\dagger a,\, \rho \right\} \\ &\quad + G(a^\dagger a)\, a^\dagger \rho a - \frac{G(a^\dagger a)}{2} \left\{ a a^\dagger,\, \rho \right\}, \end{aligned} \tag{3.32}$$

(approximate because of the nonlinear gain operator).

3.5.4 Photon statistics

(Material covered in SS 20.)

To start our analysis, let us compute the rate equations for the populations $p_n \equiv \rho_{nn}$ of finding n photons in the cavity mode. The sum of damping and gain gives the master equation

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} &= -\mathrm{i}\omega_\mathrm{c}\left[a^\dagger a,\, \rho\right] - \frac{\kappa}{2}\left\{a^\dagger a,\, \rho\right\} + \kappa\, a\rho a^\dagger \\ &\quad - \frac{G(a^\dagger a)}{2}\left\{aa^\dagger,\, \rho\right\} + G(a^\dagger a)\, a^\dagger \rho a.\end{aligned} \tag{3.33}$$

(This equation is not fully correct, since the trace of ρ is no longer conserved when $G(a^\dagger a)$ depends on the photon number operator. See Sec. 3.7 for an improved treatment.)

Taking the expectation value in the state $|n\rangle$ of the master equation (3.33), we get

$$\frac{\mathrm{d}p_n}{\mathrm{d}t} = -n\kappa\, p_n + (n+1)\kappa\, p_{n+1} - (n+1)G(n)\, p_n + nG(n-1)\, p_{n-1} \tag{3.34}$$

(Here, we have again cheated a little how to evaluate the photon number in $G(n)$, for consistency with what follows.) From the terms with a negative sign, we see that transitions leave the state $|n\rangle$ with rates $n\kappa$ and $(n+1)G(n)$, as illustrated in Fig. 3.4. Looking at the rate equation for the state $|n-1\rangle$, we see that population from state $|n\rangle$ arrives at a rate $n\kappa$. We have thus identified a first process: the cavity field loses one photon at the rate $n\kappa$. This is the expected loss process. But there is also a transition from $|n\rangle$ to $|n+1\rangle$, occurring at a rate $(n+1)G(n)$. It represents both spontaneous ("+1") and stimulated emission ("n") from the laser medium. Note that the present theory requires G to be positive (inverted medium) because transition rates are positive. The dependence of $G(n)$ on the photon number again models the gain saturation, as was the case in the semiclassical theory.

The transitions we have found are summarized in figure 3.4. We can now determine the stationary state of the laser. The probabilities p_n and

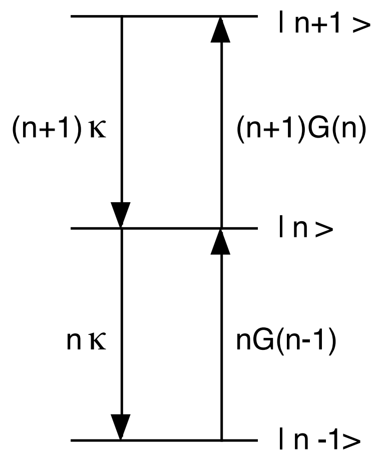


Figure 3.4: Transitions between photon number states.

p_{n+1}, say, then do not change with time, and therefore the probability current for the loss process $|n+1\rangle \to |n\rangle$ must be equal to the current for the emission process $|n\rangle \to |n+1\rangle$:

$$(n+1)\kappa p_{n+1} = (n+1)G(n)p_n \tag{3.35}$$

This condition is called '*detailed balance*' and plays an important role in statistical physics.

For the pair of levels $|n\rangle$ and $|n-1\rangle$, we get $n\kappa p_n = nG(n-1)p_{n-1}$ from the rate equation (3.34) which can also be found by shifting the label n in Eq.(3.35). We may ask whether for the transition $|0\rangle \leftrightarrow |1\rangle$, there is saturation or not: can the field of a single photon saturate the laser medium? The answer cannot come from the semiclassical model we started with because it does not know about the concept of a 'single-photon field'. But there are quantum models for a laser medium that show this saturation effect, under special conditions (typically, a high-quality laser cavity with a small volume is needed where the single-photon field is large enough).

The detailed balance equation (3.35) expresses a dynamic equilibrium between loss and gain processes; it gives a recurrence relation for the photon number probabilities in the stationary state. It is easily solved with the saturation model (3.31) to give

$$p_{n+1} = \frac{G(n)}{\kappa} p_n = \frac{G_0/\kappa}{1+Bn} p_n$$

$$\Rightarrow p_n = \mathcal{N} \prod_{m=0}^{n-1} \frac{G(m)}{\kappa} = \mathcal{N} \left(\frac{G_0}{\kappa} \right)^n \prod_{m=0}^{n-1} \frac{1}{1 + Bm}, \tag{3.36}$$

where $\mathcal{N}$ is a normalization constant. Below threshold, $G_0 < \kappa$, each of the ratios $G(n)/\kappa$ is smaller than unity, and the most probable state is the vacuum — perfectly reasonable because the laser intensity is damped away. Above threshold and for weak saturation, $G(n)/\kappa \approx G_0/\kappa > 1$, and photon numbers larger than zero are favoured. The maximum of the distribution is reached at a photon number $n_{\max}$ where $G(n_{\max})/\kappa = 1$. This equation can be solved to give

$$n_{\max} = \frac{G_0 - \kappa}{\kappa B}$$

which looks very similar to the steady state intensity of the semiclassical theory.

The photon statistics (3.36) is plotted in figure 3.5 for a laser below and above threshold. Note that below threshold, we do not have exactly a thermal state (the probability is not an exponential $\propto \mathrm{e}^{-\beta n \hbar \omega_c}$), and that above threshold, the width of the number distribution is larger than for a coherent state with the same most probable photon number. (The coherent state generates a Poisson distribution for p_n which provides a convenient reference. The photon statistics of the laser model in Fig. 3.5(right) is called 'super-Poissonian' because it is wider.)

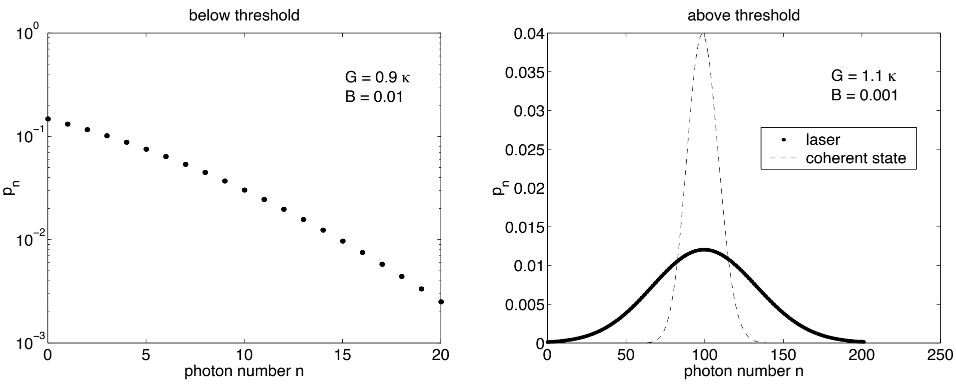


Figure 3.5: Photon statistics of a laser in steady state. Left: below threshold $G \equiv G_0 < \kappa$, right: above threshold.

Exercise. Use the following representation of the product in (3.36)

$$\prod_{m=0}^{n-1} \frac{1}{1 + Bm} = B^{-n} \frac{\Gamma(1/B)}{\Gamma(1/B + n)}$$

where $\Gamma(\cdot)$ is the gamma function to discuss the shape of the photon statistics. Using the Stirling formula for large values of n and $1/B$, show that p_n has the form of a truncated gaussian distribution and compute its width. You will find that the width approaches that of a coherent state (also known as Poisson statistics)

$$\Delta n^2 \to n_{\max}$$

when the laser is operating far above threshold. (This is difficult to achieve in practice, however.)

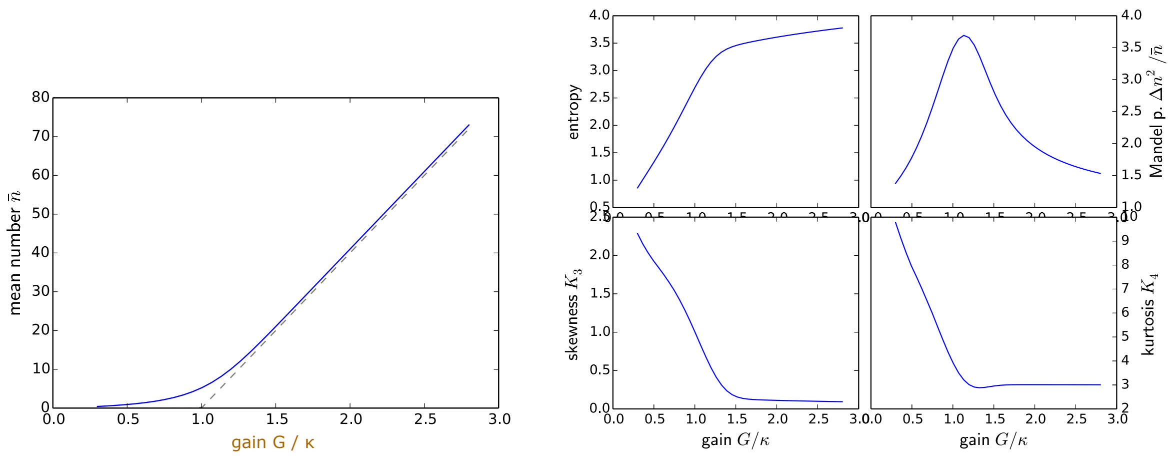


Figure 3.6: Characterisation of the photon statistics according to the Scully–Lamb model. (*left*) Average photon number vs. linear gain parameter G. (*right*) Statistical analysis and higher moments vs. G. Top left: (information) entropy of the photon statistics. Top right: Mandel parameter Q. Bottom left: skewness (third cumulant or centred third moment). Bottom right: kurtosis (fourth cumulant or centred fourth moment). Poisson statistics corresponds to $Q = 1$. Gaussian statistics corresponds to $K_4 = 3$.

In Fig. 3.6, we show the mean photon number and other statistical parameters that can be derived from the photon statistics. The mean photon number behaves similar to the semiclassical laser model (Fig. **??**), but the threshold is 'rounded' and no longer discontinuous.

The variance of the photon number $(\Delta n)^2$ is often expressed in terms of the so-called Mandel parameter

$$Q = \frac{(\Delta n)^2}{\langle \hat{n} \rangle} \tag{3.37}$$

The Mandel parameter takes the value $Q = 1$ if the photon statistics is Poissonian – this is reached well above the threshold. Below threshold, this is also true, although not very interesting since the total output intensity $\sim \langle \hat{n} \rangle$ is small. The peak of Q near the threshold is a manifestation of so-called 'critical fluctuations' – the system explores a large range of values around the mean because it is switching between two qualitatively different 'phases' (adopting the language of a phase transition).

In Fig. 3.7 below, we show the photon statistics (right) and the Mandel parameter for a specific laser model that qualitatively behaves similar to that of Scully & Lamb. (More details in Sec.3.7.)

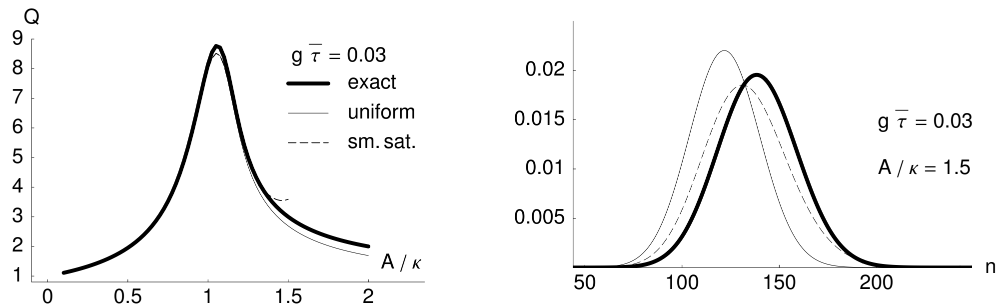


Figure 3.7: (*left*) Mandel parameter for a micromaser model, calculated with different approximations to the master equation (lines, 'uniform' = uniform approximation, 'sm. sat' = expansion for small saturation). The parameter $g\bar{\tau}$ qualitatively describes saturation, and corresponds to a parameter $B = (g\bar{\tau})^2$ in the Scully & Lamb model. The linear gain rate G_0 is denoted A in the plot, and κ is the loss rate of the laser cavity.
(*right*) Photon statistics above threshold, calculated with the same approximations. The distribution functions p_n are represented by continuous lines. Figures taken from Henkel (2007), discussed in Sec.3.7.

3.6 Fokker-Planck equation

(Material only sketched in SS 20.)

This is the name for the equation of motion of the phase-space distribution function P that represents ρ. In the quantum optics books, one learns that the P-function (or Glauber–Sudarshan function) does this job: $P(\alpha)$ gives the (quasi-)probability that a coherent state $|\alpha\rangle$ occurs in an expansion of the density operator in the basis of coherent states:[4]

$$\rho = \int \mathrm{d}^2\alpha\, P(\alpha)|\alpha\rangle\langle\alpha|. \tag{3.38}$$

3.6.1 Derivation from the master equation

What is the equation of motion for this distribution? From the quantum optics books or as an exercise, you can find the action of the photon operators on the projector $|\alpha\rangle\langle\alpha|$. This leads to the following replacement table

$$\begin{aligned} a\rho &\mapsto \alpha P \\ a^\dagger\rho &\mapsto \left(\alpha^* - \frac{\partial}{\partial\alpha}\right) P \\ \rho a &\mapsto \left(\alpha - \frac{\partial}{\partial\alpha^*}\right) P \\ \rho a^\dagger &\mapsto \alpha^* P. \end{aligned} \tag{3.39}$$

Exercise. Show that these rules are consistent with (1) the commutation relations of a and $a^\dagger$ and (2) associative operator products: consider for example, $(a^\dagger\rho)a$ and $a^\dagger(\rho a)$, that should be mapped to the same differential operator acting on P.

The equation resulting from (3.32) is

$$\begin{aligned} \frac{\partial}{\partial t}P(\alpha,\, \alpha^*) &= \frac{1}{2}\frac{\partial}{\partial\alpha}(\kappa - G)\alpha\, P + \frac{1}{2}\frac{\partial}{\partial\alpha^*}(\kappa - G)\alpha^*\, P \\ &\quad + \frac{G}{4}\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}P, \end{aligned} \tag{3.40}$$

(The derivatives act on everything to their right, including P.) We have yet neglected gain saturation: an approximate way to take it into account is to replace $G \mapsto G(|\alpha|^2) = G_0/(1 + B|\alpha|^2)$ and to put this in between the two derivatives in the diffusion term in Eq.(3.40). This involves an approximation because we do not keep the full operator dependence of the gain, actually neglecting some second-order and higher-order derivatives.

Let us note that the second-order derivative $\partial_\alpha \partial_{\alpha^*}$ in Eq.(3.40) is directly related to the fact that in the quantum description, the operators a and $a^\dagger$ do not commute. (Their

[4]The quasi-probability $P(\alpha)$ is *not* identical to the diagonal matrix element $\langle\alpha|\rho|\alpha\rangle$ which is called the Q-function (or Husimi function). The difference is that for some states, $P(\alpha)$ is not positive everywhere and may not even exist as an ordinary function.

action on a coherent state cannot reduce to multiplication with the numbers α and α^*, but must involve some derivative, as seen in the replacement rules (3.39)). We already suspect that the second-order derivative may have to do with phase diffusion. We see here that it is connected to the discrete nature of the photons. Sometimes, people develop the picture that each photon that is spontaneously emitted by the gain medium contributes to the cavity field a kind of "one-photon field" whose phase is arbitrary. The amplitude of the cavity field thus performs a "random walk" in phase space together with a "deterministic" increase related to "stimuled emission" where the additional photons add up "in phase" with the field.

We now have with (3.40) a partial differential equation for a phase space distribution. It features second-order derivatives like the simple diffusion equation (3.47). Since we are interested in phase diffusion, it seems natural to use polar coordinates $\alpha = r\,\mathrm{e}^{\mathrm{i}\phi}$. You are asked to make this transformation in the exercises and to derive the result

$$\frac{\partial}{\partial t}P(r,\,\phi) = \frac{1}{2r}\frac{\partial}{\partial r}r^2(\kappa - G(r^2))P + \frac{1}{4r}\left(\frac{\partial}{\partial r}rG(r^2)\frac{\partial}{\partial r} + \frac{G(r^2)}{r}\frac{\partial^2}{\partial\phi^2}\right)P. \tag{3.41}$$

The steady state solution of this equation does not depend on the phase, but only on the modulus r of the laser amplitude.

Let us focus on the stationary state, $\partial_t P = 0$. One can show by taking the limit $r \to \infty$ and imposing the integrability of the P-function that the constant in

$$r^2(\kappa - G(r^2))P + \frac{1}{2}rG(r^2)\frac{\partial}{\partial r}P(r) = \text{const.} = 0 \tag{3.42}$$

is actually zero. ('$P(r)$ and its derivative vanish at infinity.') It is then easy to check that the solution to Eq.(3.42) is (up to a normalization)

$$P(r) \sim \exp\left[-2\int^{r}\mathrm{d}r'\,r'\frac{G(r^2)-\kappa}{G(r^2)}\right] \tag{3.43}$$

For the gain model of Scully & Lamb [Eq.(3.31)],

$$G(r^2) = \frac{G_0}{1+Br^2}\,, \tag{3.44}$$

the integral is easy to work out (substitute $n = r^2$) and

$$P(r) \sim \exp\left[-\frac{B\kappa}{2G_0}(r^2-\bar{n})^2\right], \qquad \bar{n} = \frac{G_0-\kappa}{\kappa B} \tag{3.45}$$

This shows a maximum at $r^2_{\max} = \bar{n}$ where we recover the formula for the semiclassical steady-state intensity (up to a conversion factor between intensity and photon number). From this distribution, one can check that the fluctuations of the laser intensity are small in the limit $1/B \gg 1$ (large photon number on average, meaning high above threshold). We can thus confirm that if there are fluctuations, they occur dominantly in the phase of the field.

The diffusion part of the Fokker-Planck equation determines the width of the Gaussian around its maximum – typically it is wider than the (Poissonian) photon number distribution of a coherent state. The Mandel parameter [see Eq.(3.37)] has been introduced to quantify the deviation from Poisson statistics. The diffusion coefficient also determines how the phase of the laser is drifting (better: diffusing), a process that gives an ultimate limit to the width of the laser frequency spectrum (Schawlow-Townes limit). For details, see Sec.3.6.2.

3.6.2 Schawlow–Townes linewidth of the laser

Idea: phase diffusion

The spectrum of the laser is related to the autocorrelation function of the laser mode operator a:

$$S(\omega) = \lim_{t\to\infty} \int \mathrm{d}\tau \, \mathrm{e}^{-\mathrm{i}\omega\tau} \langle a^\dagger(t+\tau) a(t) \rangle \tag{3.46}$$

where the average is taken in the stationary state that is reached at large times. We shall denote this state $\rho_{\rm ss}$ or by the subscript $\langle \ldots \rangle_{\rm ss}$. If the mode were evolving freely at the laser frequency, $a(t+\tau) = a(t)\,\mathrm{e}^{-\mathrm{i}\omega_{\rm L} t}$, then the spectrum would be a δ-peak, $S(\omega) = \langle a^\dagger a \rangle_{\rm ss} 2\pi\delta(\omega - \omega_{\rm L})$ and proportional to the mean photon number. This is no longer true when we take into account that the gain mechanism also involves "spontaneous emission" where atoms of the pumping medium emit a spontaneous photon (not stimulated, hence no phase relation with the existing laser field). This gives the laser field a "fluctuating amplitude" that we have to characterize.

The main idea of Schawlow & Townes is that the laser field is essentially subject to phase fluctuation, but not to intensity fluctuations. In a semiclassical description, we thus have a mode amplitude $a(t) = \sqrt{\bar{n}_{\rm ss}}\,\mathrm{e}^{\mathrm{i}\phi(t) - \mathrm{i}\omega_{\rm L} t}$, where only the phase is fluctuating. We shall see below that the fluctuations of the laser phase are "diffusive" – the phase makes a "random walk". In terms of a distribution function $P(\phi, t)$, this behaviour is described by a diffusion equation,

$$\frac{\partial}{\partial t} P = D \frac{\partial^2}{\partial \phi^2} P \tag{3.47}$$

where D is called "phase diffusion coefficient". You may remember this equation from heat conduction. Its solution, for an initial state with a fixed phase ϕ_0, is given by a gaussian distribution

$$P(\phi, t|\phi_0) = \frac{1}{\sqrt{4\pi D t}} \mathrm{e}^{-(\phi-\phi_0)^2/4Dt}, \qquad t > 0. \tag{3.48}$$

The width of the gaussian is $2Dt$ and increases with time. For very large times, the distribution is completely flat. (This solution neglects the fact that the phase is only defined in the interval $[0, 2\pi]$. See the exercises for this case.) For $t \to 0$, one recovers a δ-function centered at ϕ_0.

With this result, we can compute the temporal correlation function of the laser field,

$$\langle a^\dagger(t+\tau) a(t) \rangle = \bar{n}_{\rm ss}\,\mathrm{e}^{\mathrm{i}\omega_{\rm L}\tau} \langle \mathrm{e}^{-\mathrm{i}[\phi(t+\tau) - \phi(t)]} \rangle,$$

which gives us the spectrum by a Fourier transform with respect to τ. We can assume that the initial phase $\phi(t)$ has some value ϕ_0 and that the later phase $\phi = \phi(t+\tau)$ is a random variable with the distribution $P(\phi, \tau|\phi_0)$ of Eq.(3.48). Taking the average, we evaluate one Gaussian integral and get

$$\langle a^\dagger(t+\tau)a(t)\rangle = \bar{n}_{\rm ss}\mathrm{e}^{\mathrm{i}\omega_L\tau}\int \mathrm{d}\phi\, P(\phi,\tau|\phi_0)\mathrm{e}^{-\mathrm{i}(\phi-\phi_0)} = \bar{n}_{\rm ss}\mathrm{e}^{\mathrm{i}\omega_L\tau}\mathrm{e}^{-D\tau}.$$

The temporal correlation function thus decays exponentially with a coherence time $\tau_c = 1/D$. This argument applies only for $\tau > 0$. But if the laser state is stationary, we can argue for $\tau < 0$ that

$$\langle a^\dagger(t+\tau)a(t)\rangle = \langle a^\dagger(t)a(t-\tau)\rangle = \langle a^\dagger(t-\tau)a(t)\rangle^* \tag{3.49}$$

by taking the hermitean conjugate of the operator product. The spectrum (3.46) is thus obtained from a "half-sided Fourier transform" which is calculated as

$$S(\omega) = 2\,\mathrm{Re}\int_0^\infty \mathrm{d}\tau\, \mathrm{e}^{-\mathrm{i}\omega\tau}\langle E^*(t+\tau)E(t)\rangle = \frac{2I_{\rm ss}D}{(\omega-\omega_L)^2 + D^2} \tag{3.50}$$

The laser spectrum is centered at ω_L with a "Lorentzian" lineshape and a width of the order of D. *The laser linewidth is thus limited by the phase diffusion coefficient.*

3.6.3 Regression recipe for correlations

We now have to find an expression for the phase diffusion coefficient. To this end, we shall derive an equation similar to the diffusion equation (3.47). Since this equation deals with a distribution function for the phase, it seems natural to re-use the equation of motion for the (quasi-)probability distribution for the laser amplitude that we found in Sec.3.6.

Let us recall the steady state solution (3.45) of this equation: it does not depend on the phase, but only on the modulus r of the laser amplitude.

$$P_{\rm ss}(r) = \mathcal{N}\exp\left[-\frac{B\kappa}{2G_0}(r^2-\bar{n})^2\right], \qquad \bar{n} = \frac{G_0-\kappa}{\kappa B} \tag{3.51}$$

Recall that the fluctuations of the laser intensity (and the field's modulus) are small in the limit $1/B \gg 1$ (large photon number on average, meaning high above threshold). We can thus confirm that if there are fluctuations, they occur dominantly in the phase of the field.

The spectrum of the laser is determined by the autocorrelation function of the field operator. For simplicity, we consider the stationary state where one expects that it depends only on the time difference

$$C(\tau) = \lim_{t\to\infty}\langle a^\dagger(t+\tau)a(t)\rangle \tag{3.52}$$

To describe such a correlation, we invoke the following 'two measurement scheme' – which is nothing but a heuristic translation of something that is quantum-mechanically quite subtle. In the literature, this is called the 'regression theorem' (although it is not a theorem).

Let us imagine that at time t, we make a measurement of $a(t)$. The expectation value would be given by (we are in the stationary state)

$$\langle a(t)\rangle = \int \mathrm{d}^2\alpha\, \alpha\, P_{\rm ss}(\alpha) \tag{3.53}$$

But we are actually interested in what happens until time $t + \tau$ where the second measurement is made. From the Fokker-Planck equation, we can construct a 'conditional' distribution, $P(\alpha', \tau|\alpha)$: the P-function for a laser that starts at time t in a coherent state α. It is tempting to assume that this function solves the same Fokker-Planck equation as $P_{\rm ss}$, but (i) including the time-dependence and (ii) with the initial condition

$$P(\alpha', \tau = 0|\alpha) = \delta(\alpha' - \alpha) \tag{3.54}$$

Such a function is called the 'Green function' of the 'Fokker-Planck operator. If we know it (and our reasoning is correct), then we can calculate the two-time correlation (3.52) in the form

$$\langle a^\dagger(t + \tau)a(t)\rangle = \int \mathrm{d}^2\alpha' \mathrm{d}^2\alpha\, \alpha'^* P(\alpha', \tau|\alpha)\alpha P(\alpha, t) \tag{3.55}$$

where we made the generalisation to a not-yet-steady state.

If we know imagine that the first distribution $P(\alpha, t) = P_{\rm ss}(\alpha)$ is sharply peaked in the radial direction, we may restrict the integration $\mathrm{d}^2\alpha$ to the phase of the field amplitude only. In this regime, we recover a pure phase diffusion process. The 'Green function' takes this coherent state (with fixed photon number, but random phase) and since this is a localised state that is narrower than the equilibrium distribution, the diffusion term in the Fokker-Planck equation broadens it. In the radial direction, this broadening is rapidly stopped because of the competing effects of gain and loss. But 'along the circle', the distribution can diffuse freely.

With this argument, we can go back to the diffusion equation (3.41) and identify the phase diffusion coefficient as the prefactor of the ∂_ϕ^2 derivative:

$$D = \frac{G(r^2)}{4r^2} \approx \frac{G(n_{\rm max})}{4n_{\rm max}}. \tag{3.56}$$

This formula has been derived first in the 1960/70's by Schawlow and Townes. It shows that phase diffusion (and the laser linewidth), well above threshold, decreases inversely proportional to the laser intensity. When the gain is increased, the emission spectrum thus shows an ever growing peak close to the frequency of the cavity mode, that becomes narrower and narrower. This behaviour is often taken as an experimental proof that a laser is operating.

3.6.4 Transient dynamics

Elements can be found in Chapters 18.6–8 in Mandel & Wolf (1995):
– Solution of time-dependent equation of motion
– Correlation functions
– Laser instabilities and chaos

3.7 Micromaser model

3.7.1 Motivation

(Material taken from "Laser theory in manifest Lindblad form" (Henkel, 2007). Not covered in SS 2019.)

The quantum theory of a laser is a textbook example of a nonlinear problem that requires techniques from open quantum systems. The key issue is the nonlinearity in the gain of the laser medium, due to saturation, that leads to coupled nonlinear equations already at the semiclassical level. The quantum theory makes things worse by its use of non-commuting operators.

Recall that in the so-called semiclassical theory (see Sec.3.4), the following equation of motion for the intensity I the laser mode can be derived (Sargent III & Scully, 1972; Orszag, 2000):

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -\kappa I + \frac{GI}{1+\beta I} \tag{3.57}$$

where κ is the loss rate, G is the linear gain, and β describes gain saturation for the laser medium. A quantum upgrade of this theory replaces the intensity by the photon number $a^\dagger a$ where the annihilation operator a describes the field amplitude of the laser mode. Mode loss is easy to handle by coupling the laser mode linearly to a mode continuum 'outside' the laser cavity (Walls & Milburn, 1994). This leads to a master equation for the density matrix in so-called Lindblad form [see Eq.(3.28)]

$$\left.\frac{\mathrm{d}\rho}{\mathrm{d}t}\right|_{\mathrm{loss}} = (L\rho L^\dagger - \tfrac{1}{2}\{L^\dagger L,\, \rho\}) \tag{3.58}$$

with a Lindblad operator $L = L_{\mathrm{loss}} = \sqrt{\kappa}\, a$. Linear gain can be handled in the same way, taking $L_{\mathrm{gain},0} = \sqrt{G}\, a^\dagger$, but gain saturation is more tricky. A heuristic conjecture is a Lindblad operator $L_{\mathrm{gain}} = \sqrt{G}\, a^\dagger(1 + \beta a^\dagger a)^{-1/2}$. The operator ordering can only be ascertained *a posteriori*, and it is difficult to choose among the replacements $I \mapsto a^\dagger a$, $aa^\dagger$, or $\frac{1}{2}\{a^\dagger a + aa^\dagger\}$. We illustrate this difficulty in Sec.3.5 where the conventional quantum theory of the laser (due to Scully & Lamb) is presented.

In this section, we start with a microscopic model for the pumping process. This is motivated by experiments with so-called micromasers where a (microwave) cavity is crossed by a beam of excited two-level atoms. Nonlinear gain emerges from a treatment beyond second order in the atom-field coupling. Orszag (2000); Stenholm (1973) analyze a pumping model based on a dilute stream of excited two-level atoms that cross the laser cavity one by one and interact with the laser mode during some randomly distributed interaction time. This model can be largely handled exactly (Briegel & Englert, 1993), even in the presence of incoherent effects like cavity damping, imperfect atom preparation, and frequency-shifting collisions. The setup has become known as the 'micromaser' because of its experimental realization with a high-quality cavity (Meschede & al., 1985; Brune & al., 1987; Raizen & al., 1989). One line of research has focused on the so-called 'strong coupling regime' that permits the laser mode to be driven into non-classical states (Weidinger & al., 1999; Varcoe & al., 2000).

We focus here on the 'weak coupling' regime. On the level of the master equation for the laser mode, this regime corresponds to a small product of coupling constant and elementary interaction time τ so that one can expand in this parameter. Mandel & Wolf (1995); Orszag (2000) consider a coupling to fourth order. For the description of a realistic experiment, one has to average the master equation with respect to a distribution of the interaction time τ (Sec. 3.7). It turns out, however, that the resulting master equation is not of the well-known Lindblad form, although it preserves the trace of the density matrix. This leads to conflicts with the positivity of the density operator, as is known since the original derivation of the master equation by Lindblad and by Gorini et al. (Lindblad, 1976; Gorini & al., 1976). We have shown that this problem can be cured by adding certain terms in sixth order to the master equation (Henkel, 2007). The material presented here is based on this reference. This problem provides us with an example where the Lindblad master equation can be derived from the Kraus-Stinespring representation of the finite-time evolution of the density matrix. The mathematical treatment is at the border of validity of the formal Lindblad theory since one has to deal with an infinite-dimensional Hilbert space and continuous sets of Kraus and Lindblad operators.

3.7.2 The micromaser model

Consider a two-level atom with states $|g\rangle$, $|e\rangle$ that is prepared at time t in its excited state $|e\rangle = (1, 0)^T$ (density matrix $\rho_A = |e\rangle\langle e|$) and that interacts with a single mode (density matrix ρ) during a time τ. One adopts a Jaynes-Cummings-Paul Hamiltonian for the atom-field coupling

$$H_{\rm JCP} = \hbar g \left(a^\dagger \sigma + a \sigma^\dagger \right), \qquad \sigma = |g\rangle\langle e| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{3.59}$$

(this applies at resonance in a suitable interaction picture). Assume that the initial density operator of the atom+field-system factorizes into $P(t) = \rho(t) \otimes \rho_A$, compute $P(t+\tau)$ by solving the Schrödinger equation and get the following reduced field density matrix (Orszag, 2000; Stenholm, 1973)

$$\rho(t+\tau) = \cos(g\tau\hat{\varphi})\rho(t)\cos(g\tau\hat{\varphi}) + (g\tau)^2 a^\dagger \operatorname{sinc}(g\tau\hat{\varphi})\rho(t)\operatorname{sinc}(g\tau\hat{\varphi})a \qquad (3.60)$$

where $\operatorname{sinc}(x) \equiv \sin(x)/x$, and $\hat{\varphi}^2 = aa^\dagger$ is one plus the photon number operator. The operator-valued functions $\cos$ and sinc are defined by their series expansion. Only even powers of the argument occur, hence we actually never face the square root $\hat{\varphi}$ of the operator $aa^\dagger$. In the following, we abbreviate the mapping defined by Eq.(3.60) by $\mathbb{M}_\tau\rho(t)$ (this is sometimes called a superoperator).

The operation (3.60) describes an elementary 'pumping event' of the laser. We assume in the following that this event provides only a small change in the density operator, $(\mathbb{M}_\tau - \mathbb{1})\rho$ is 'small'. To provide a more realistic description, one introduces the following additional averages: excited atoms appear in the laser cavity at a rate r such that $r\tau \ll 1$. Over a time interval Δt, a number of $r\Delta t$ pumping events happens, and the accumulated change in the density operator is $\Delta\rho = r\Delta t(\mathbb{M}_\tau - \mathbb{1})\rho$. This will lead us to a differential equation when Δt is 'small enough'.

We make the additional assumption that the interaction time τ is distributed according to the probability measure $\mathrm{d}p(\tau)$ with mean value $\bar{\tau}$. This reflects the fact that atoms can cross the cavity mode with different velocities, at different positions etc. Also in a conventional laser, atoms interact with the laser mode only during some finite (and randomly distributed) time of the order of the lifetime of the excited state. Keeping the coarse-grained time step ($\Delta t \gg \bar{\tau}$), we thus get the difference equation (Orszag, 2000; Stenholm, 1973)

$$\frac{\Delta\rho}{\Delta t} = r\int \mathrm{d}p(\tau)\ (\mathbb{M}_\tau - \mathbb{1})\,\rho. \qquad (3.61)$$

To simplify the superoperator appearing on the right hand side, Orszag (2000); Stenholm (1973) suggest an expansion in powers of $g\tau\hat{\varphi}$ up to the fourth order. Using an exponential distribution for $\mathrm{d}p(\tau)$, this leads to the approximate master equation

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} &= G\left(a^\dagger\rho a - \tfrac{1}{2}\{aa^\dagger,\,\rho\}\right) \\ &\quad + \mathcal{B}\left(3aa^\dagger\rho aa^\dagger + \tfrac{1}{2}\{(aa^\dagger)^2,\,\rho\} - 2a^\dagger\{aa^\dagger,\,\rho\}a\right) \qquad (3.62)\end{aligned}$$

where we followed the common practice of interpreting this as a differential equation. We use $\{\cdot,\cdot\}$ to denote the anticommutator. The linear gain is $G = 2r(g\bar{\tau})^2$, and $\mathcal{B} = (g\bar{\tau})^2 G$ is a measure of gain saturation. Indeed, the first line of Eq.(3.62) is in Lindblad form with $L_{\rm gain} = \sqrt{G}\, a^\dagger$ – applying this operator *increases* the photon number by one. It is easy to see that this leads, on average, to an increasing field amplitude, $({\rm d}/{\rm d}t)\langle a \rangle_{\rm gain} = G\langle a \rangle$.

Losses from the laser mode can be included in the usual way by adding a term of the same structure as the first line of Eq.(3.62), but featuring the Lindblad operator $L_{\rm loss} = \sqrt{\kappa}\, a$ with the cavity decay rate κ, see Eq.(3.58) (Orszag, 2000; Stenholm, 1973). The same master equation as Eq.(3.62) is also found, using a different pumping model (Mandel & Wolf, 1995).

It is easy to check that Eq.(3.62) preserves the trace of ρ, using cyclic permutations. Nevertheless, it is not of the general form derived by Lindblad for master equations that preserve the complete positivity of density matrices (Lindblad, 1976; Gorini & al., 1976; Alicki & Lendi, 1987). One can show indeed that Eq.(3.62) leads to a density matrix with negative diagonal elements (unphysical for probabilities). Of course, one can accept to work with this kind of 'post-Lindblad' master equations (as they appear frequently in the papers of Golubev and co-workers, see e.g. Golubev & Gorbachev (1986)). It is also possible to construct a set of Lindblad operators $\{L_\lambda\}$ such that with a few additional terms to the master equation (3.62), it can be brought into the Lindblad form. For more details, see "Laser theory in manifest Lindblad form" (Henkel, 2007).

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