Introduction to Quantum Optics

Winter term 2006/07
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Problem set 3 Hand out: 09 Nov 2006
Hand in: 23 Nov 2006

Problem 3.1 – Q-function and coherent states (10 points)

We deal here with the quantum states for a single mode of the quantized field.

(1) Compute the Q-function for a thermal state of the mode.

(2) Compute the overlap between two coherent states $|\alpha\rangle$ and $|\beta\rangle$ and check that the Q-function for a coherent state is an isotropic gaussian in the complex α-plane.

(3) The coherent states are not orthogonal, as the result of question (2) shows. Hence it is not surprising that they are 'overcomplete' in the following sense:

$$\int \mathrm{d}^2\alpha \, |\alpha\rangle\langle\alpha| = \pi \mathbb{1} \tag{3.1}$$

where the unit operator is for example $\mathbb{1} = \sum_{n=0}^{\infty} |n\rangle\langle n|$. Prove this equation, using the standard integration measure in the complex plane: $\mathrm{d}^2\alpha = \mathrm{d}(\mathrm{Re}\,\alpha)\,\mathrm{d}(\mathrm{Im}\,\alpha)$. The integral given in Problem 3.3 should be useful.

(4) Derive the following identity for an arbitrary density operator ρ

$$\langle a^m a^{\dagger\, n} \rangle_\rho = \int \frac{\mathrm{d}^2\alpha}{\pi} \alpha^m \alpha^{*\, n} Q_\rho(\alpha) \tag{3.2}$$

Speculate why this is usually interpreted as: *For averages in 'anti-normal order', the Q-function plays the role of a probability distribution.*

Problem 3.2 – Casimir repulsion (10* points)

Revisit the calculation of the Casimir energy that we did in the lecture for mixed 'Dirichlet-von Neumann' boundary conditions.

(2) Justify that the allowed wave vectors along the cavity axis are given by $k_n L = \pi(n + \frac{1}{2})$ with $n = 0, 1, 2, \ldots$.

(3) Perform the calculation of the Casimir energy in the same way as in the lecture. You will need the integral

$$\int_0^\infty \frac{\mathrm{d}t\, t^3}{\mathrm{e}^{2\pi t} + 1} = \frac{7}{1920} \tag{3.3}$$

(Bonus points.)

Problem 3.3 – Dipole matrix elements (10 points)

Consider the energy levels $nl = 1\mathrm{s}$, $2\mathrm{s}$ and $2\mathrm{p}$ of the Hydrogen atom. The coupling to the light field involves in the long-wavelength approximation the following matrix elements of the electron coordinate $\mathbf{r}$ (measured relative to the nucleus):

$$-e\langle nlm|\mathbf{r}|n'l'm'\rangle = -e\int \mathrm{d}^3r\,\psi^*_{nlm}(\mathbf{r})\,\mathbf{r}\,\psi_{n'l'm'}(\mathbf{r}). \tag{3.4}$$

The atom can absorb a photon on the $nl \to n'l'$ transition if this matrix element is nonzero.

(1) Show that these matrix elements are zero if l and l' have the same parity.

(2) Compute the matrix elements between the $1\mathrm{s}$ and $2\mathrm{p}$ states.

Hydrogen wave functions. They are of the form (Messiah, vol. 1, appendix B)

$$\psi_{nlm}(r,\theta,\varphi) = a_0^{-3/2} R_{nl}(r) Y_l^m(\theta,\varphi)$$

with the Bohr radius a_0, the radial wave function R_{nl} and the spherical harmonic (*Kugelflächenfunktion*) Y_l^m. For the states in question, the radial functions are given by

$$\begin{aligned}
&\text{1s:} && R_{10}(r) = 2\mathrm{e}^{-r/a_0} \\
&\text{2s:} && R_{20}(r) = \frac{\sqrt{2}}{2}\left(1 - \frac{r}{2a_0}\right)\mathrm{e}^{-r/2a_0} \\
&\text{2p:} && R_{21}(r) = \frac{\sqrt{6}\,r}{12\,a_0}\mathrm{e}^{-r/2a_0}
\end{aligned}$$

The first few spherical harmonics are given by (Messiah's phase convention)

$$\begin{aligned}
\text{s orbital:}\quad & Y_0^0(\theta,\varphi) &=& \frac{1}{\sqrt{4\pi}} \\
\text{p orbitals:}\quad & Y_1^0(\theta,\varphi) &=& \sqrt{\frac{3}{4\pi}}\cos\theta \\
& Y_1^{+1}(\theta,\varphi) &=& -\sqrt{\frac{3}{8\pi}}\sin\theta\,\mathrm{e}^{\mathrm{i}\varphi} \\
& Y_1^{-1}(\theta,\varphi) &=& \sqrt{\frac{3}{8\pi}}\sin\theta\,\mathrm{e}^{-\mathrm{i}\varphi}
\end{aligned}$$

For both questions, it is useful to compute first the integrals over the angles θ and φ. For the radial integrals, you can use

$$\int_0^\infty \mathrm{d}x\,x^n\mathrm{e}^{-x} = \Gamma(n+1) = n!$$