Introduction to Quantum Optics

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Problem set 5 Hand out: 08 Dec 2006
Hand in: 21 Dec 2006

Problem 5.1 – Two-level density operator (4 points)
Show that the Pauli matrices σ_i ($i = 1, 2, 3$) satisfy the relation $\mathrm{tr}(\sigma_i \sigma_j) = 2\delta_{ij}$ and conclude that the following representation of the density operator for a two-level system is true

$$\rho = \frac{1}{2}\left(\mathbb{1} + \sum_i \sigma_i s_i\right) \tag{5.1}$$

where s_i is the Bloch vector.

Problem 5.2 – Jaynes-Cummings-Paul model (6 points)
Consider the following Hamiltonian

$$H = \hbar\omega_{eg}\sigma^\dagger\sigma + \hbar\omega a^\dagger a + \hbar g\left(a^\dagger\sigma + \sigma^\dagger a\right) \tag{5.2}$$

(i) Describe in words the physical interpretation of the three terms. What approximation has been made here?

(ii) Show that the following quantity, called 'excitation', is conserved under the Hamiltonian (5.2):

$$N = \sigma^\dagger\sigma + a^\dagger a \tag{5.3}$$

and speculate why this is so.

(iii) In the subspace of fixed excitation number $N > 0$, show that the Hamiltonian (5.2) has the energy eigenvalues

$$E_N^{(\pm)} = \text{const.} \pm \frac{\hbar}{2}\sqrt{(\omega_{eg} - \omega)^2 + g^2 N} \tag{5.4}$$

and determine the value of the constant (it may depend on N). What happens for $N = 0$?

Problem 5.3 – Spontaneous emission (10 points)
As a variant to the lecture, consider the problem of spontaneous emission in the Schrödinger picture. The starting point is the Hamiltonian

$$H = \hbar\omega_{eg}\sigma^\dagger\sigma + \sum_k \hbar\omega_k a_k^\dagger a_k + \sum_k \hbar g_k\left(a_k^\dagger\sigma + \sigma^\dagger a_k\right) \tag{5.5}$$

with the usual commutation relations and the 'spectral density' in the vacuum state

$$\sum_k g_k^2 \delta(\omega - \omega_k) = S(\omega) \tag{5.6}$$

where $S(\omega)$ is a smooth, real function.

(i) Consider the initial state $|\psi(0)\rangle = |e\rangle \otimes |\text{vac}\rangle$ and show that the Ansatz

$$|\psi(t)\rangle = c_e(t)|e\rangle \otimes |\text{vac}\rangle + c_k(t)|g\rangle \otimes |1_k\rangle \tag{5.7}$$

solves the Schrödinger equation for $t > 0$. ($|1_k\rangle$ is the state with one photon in the mode k.) Write down the equations for the amplitudes c_e, c_k.

(ii) Eliminate the amplitudes c_k from the equations and derive a closed equation for c_e of the form

$$i\dot{c}_e = \omega_{eg} c_e + \int_0^t \mathrm{d}t' \, \chi(t - t') c_e(t') \tag{5.8}$$

Determine the relation between the response function $\chi(\tau)$ and the spectral density $S(\omega)$.

(iii) Solve Eq.(5.8) with a Laplace transformation within the 'so-called simple pole approximation', $\chi(\tau) = \omega_c \chi_0 \, \mathrm{e}^{-\omega_c \tau}$. To simplify the calculation, take the limit $\omega_c \to \infty$. Note that this is equivalent to replacing the Laplace transform of χ by a constant. You get an exponential decay for $|c_e(t)|^2$ with a rate proportional to the imaginary part of χ_0. (Bonus points for the analysis of the Laplace transform in the general case.)