

2.6 Light detectors and coherence

2.6.1 Glauber photodetector theory

We now have to look in more detail at how exactly the quantum state of the electromagnetic field can be measured. The devices that measure light are called “photodetectors”. We shall sketch here the theory that R. Glauber developed to describe these detectors. Light detection has some particular properties: for example, it is practically impossible to resolve the oscillations of the electric field at an optical period of about 10^{-15} s — detectors are simply “too slow” for that. They have a “response time” that is typically in the 10^{-9} s range (“fast” photodetector). In practice (and also in the classical theory), one therefore has only access to time-averaged quantities, the average being taken over at least one optical period.

A good “quantum” model for a light detector is a two-level atom initially in its ground state that gets excited when it is illuminated by light. If we can measure the population of its excited state, we have a signal which is related to the light field. This is the starting point of Glauber’s theory.

Detector model. The details are the following: an efficient detection is possible when the two-level system is ionized by the light because the ejected electron can efficiently be detected (this is used in photomultipliers). This means, however, that our excited state is actually a continuum (the electron moves freely), and we cannot use a simple-minded two-state model. So let us assume the level scheme shown in figure 2.2 with a ground state $|g\rangle$ and a continuum C of excited states $|e\rangle$.

We are interested in the probability $p_C(t)$ that this detector goes into any of the excited states $|e\rangle$ (“detection probability”). In terms of the probability amplitudes $c_{e,g}(t)$, the detection probability is given by

$$p_C(t) = \sum_{e \in C} |c_e(t)|^2. \quad (2.49)$$

We start at $t = 0$ with the detector in the ground state. To compute the $c_e(t)$, nothing better can we do than perturbation theory, as we did in chapter 1. This gives

$$c_e(t) = -\frac{i}{\hbar} \int_0^t d\tau \langle e; f | V(\tau) | g; i \rangle e^{i\omega_{eg}\tau} \quad (2.50)$$

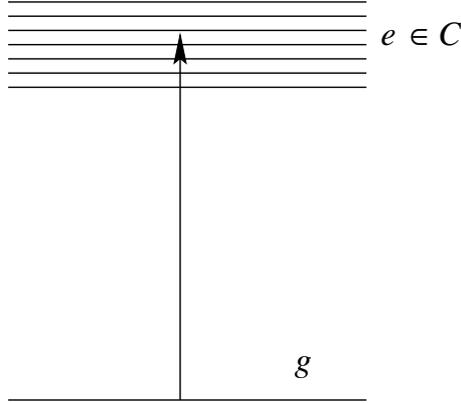


Figure 2.2: Two-level model for a photodetector.

where $|i\rangle$ and $|f\rangle$ are the initial and final field states. The interaction potential is taken in the electric dipole coupling

$$V(\tau) = -\hat{\mathbf{d}} \cdot \mathbf{E}(\tau) = -\hat{\mathbf{d}} \cdot \mathbf{E}^+(\tau) - \hat{\mathbf{d}} \cdot \mathbf{E}^-(\tau) \quad (2.51)$$

where $\mathbf{E}(\tau)$ is the electric field operator at the location of the detector, and $\mathbf{E}^\pm(\tau)$ its positive and negative frequency part: in the quantized description, the positive frequency part $\mathbf{E}^+(\tau)$ only contains annihilation operators $a_{\mathbf{k}}(\tau) \propto e^{-i\omega_{\mathbf{k}}\tau}$, and the negative frequency part only creation operators $a_{\mathbf{k}}^\dagger(\tau) \propto e^{i\omega_{\mathbf{k}}\tau}$.

Detection probability. Also in the quantum theory, we can make the resonance approximation (or equivalently, the r.w.a.) and keep in the time-integral (2.50) only those terms that lead to resonant denominators. Since the transition frequency ω_{eg} is positive, this amounts to keeping only the positive frequency part of the electric field operator. We thus have

$$c_e(t) = \frac{i}{\hbar} \mathbf{d} \cdot \int_0^t d\tau \langle f | \mathbf{E}^+(\tau) | i \rangle e^{i\omega_{eg}\tau} \quad (2.52)$$

where we have used the abbreviation $\mathbf{d} = \langle e | \hat{\mathbf{d}} | g \rangle$ and where the initial and final field states $|i, f\rangle$ have been spelled out. For the probability $p_C(t)$, we get an expression involving $\langle i | E_j^-(\tau') | f \rangle \langle f | E_i^+(\tau) | i \rangle$. At this point, we can perform the sum over the final field states $|f\rangle$ (these are not resolved, so

we add the corresponding probabilities). We thus get

$$p_C(t) = \frac{1}{\hbar^2} \sum_{e \in C} d^{j*} d^i \int_0^t d\tau d\tau' \langle i | E_j^-(\tau') E_i^+(\tau) | i \rangle e^{i\omega_{eg}(\tau - \tau')} \quad (2.53)$$

An explicit calculation of the sum over the continuum E is only possible with a specific model. Let us assume here the following form

$$\sum_{e \in C} d^{j*} d^i e^{i\omega_e \tau} = D \delta^{ij} \delta(\tau). \quad (2.54)$$

This form is plausible when the width $\Delta\omega_e$ of the continuum is sufficiently broad: for τ of the order of the time resolution of the detector, the phase factors $e^{i\omega_e \tau}$ then rapidly average out. Note that this behaviour is a consequence of the fact that “photodetectors are slow”. The Kronecker δ^{ij} for the field components is plausible when the detector does not discriminate the field polarization. This is a simplifying assumption that we relax later to describe polarization-sensitive detectors.

Putting (2.54) into the detection probability (2.53), we get the basic result

$$p_C(t) = \eta \int_0^t d\tau \langle \mathbf{E}^-(\tau) \cdot \mathbf{E}^+(\tau) \rangle, \quad \eta = \frac{D}{\hbar^2}, \quad (2.55)$$

where we see that a photodetector is sensitive to a “normally ordered” product of electric field operators (annihilation operators to the right and act first). The average $\langle \dots \rangle$ is an average over the field state — which can be a quantum average (in a pure quantum state $|i\rangle$) or a combined quantum and classical average. If the field is described by the density operator $\hat{\rho}$, then we obviously have

$$\langle \mathbf{E}^-(\tau) \cdot \mathbf{E}^+(\tau) \rangle = \text{tr} \left(\hat{\rho} \mathbf{E}^-(\tau) \cdot \mathbf{E}^+(\tau) \right).$$

The time-dependence of the operators reminds us that we deal with Heisenberg operators – the density matrix thus specifies the “initial” state (that does not evolve in time).

Photocurrent. The time derivative of the detection probability gives the production rate of photoelectrons, or (up to a factor e) the photocurrent $j(\mathbf{r}, t)$:

$$\begin{aligned} j(\mathbf{r}, t) &= \eta \langle I(\mathbf{r}, t) \rangle, \\ I(\mathbf{r}, t) &= : \mathbf{E}^-(\mathbf{r}, \tau) \cdot \mathbf{E}^+(\mathbf{r}, \tau) : \end{aligned} \quad (2.56)$$

The notation $: \dots :$ means “normal order” (creation operators to the left, annihilation operators to the right and act first). Up to this operator ordering prescription, we have found an expression that resembles the classical time-averaged Poynting vector (recall that it is also proportional to the squared complex field amplitude). Note that the average here is more involved than in the classical theory, since it also has quantum aspects.² The proportionality factor η depends on the detector, and theoretical results are often normalized to make it disappear.

If the field is a single plane wave mode, we see that the photocurrent is proportional to the photon number operator:

$$j(t) = \eta_1 \langle a^\dagger(t)a(t) \rangle \quad (2.57)$$

with $\eta_1 = \eta E_{\text{1ph}}^2$. In this context, the factor η_1 is often interpreted as “detection efficiency”, i.e., the probability that a photon is detected.

Polarization filtering

Imagine a polarization filter in front of the photodetector. In classical terms, this means that only a component $\mathbf{u}^* \cdot \mathbf{E}$ is transmitted to the detector.³ In the quantum theory, we can describe this by expanding the field $\mathbf{E}(\mathbf{r}, t)$ into modes polarized parallel and perpendicular to \mathbf{u} . Only the mode with parallel polarization has a mode function that reaches the detector (recall that the calculation of mode functions is a classical problem, independent of the field quantization). So our photocurrent becomes

$$j_{\mathbf{u}}(\mathbf{r}, t) = \langle \mathbf{u} \cdot \mathbf{E}^-(\mathbf{r}, t) \mathbf{u}^* \cdot \mathbf{E}^+(\mathbf{r}, t) \rangle$$

Frequency spectrum

What happens with filtering in frequency space? We can in fact use a very similar reasoning: a classical mode at frequency ω will be transmitted to

²A more precise theory is possible at the classical level (“optical coherence theory”): there, the light field is treated as a stochastic field with fluctuations, since it often originates from many uncorrelated sources (classical representation of individual atoms). In the theory, an extra average over these fluctuations then enters. Experimentally, this average enters because detectors are “slow” and cannot resolve these fluctuations.

³The complex conjugate \mathbf{u}^* is applicable when a filter for circular polarization is used.

the detector with an efficiency given by the transmission coefficient of the filter. This quantity can be calculated for a cavity, e.g., and shows peaks (maximum transmission) at specific resonance frequencies. Let us focus on one of these frequencies, say ω_c , and suppose that the filter has a very narrow resonance. The transmitted field is then proportional to the time-Fourier transformed electric field at the resonance frequency ω_c :

$$\mathbf{E}_{\text{tr}}^+(t) \propto \tilde{\mathbf{E}}^+(\omega_c) e^{-i\omega_c t}$$

The corresponding photocurrent is then given by

$$\begin{aligned} j_{\omega_c}(t) &\propto \langle \tilde{\mathbf{E}}^-(\omega_c) \cdot \tilde{\mathbf{E}}^+(\omega_c) \rangle \\ &= \int dt_1 dt_2 e^{-i\omega_c(t_1-t_2)} \langle \mathbf{E}^-(t_1) \cdot \mathbf{E}^+(t_2) \rangle \end{aligned} \quad (2.58)$$

If we assume that the field is stationary in time, we can shift the time arguments in the product of field averages and write

$$j_{\omega_c}(t) \propto T \int d\tau e^{i\omega_c \tau} \langle \mathbf{E}^-(0) \cdot \mathbf{E}^+(\tau) \rangle \quad (2.59)$$

where T is the measurement time (at least equal to the time needed to build up the field in the cavity). This equation can be generalized to any detector (“spectrometer”) that measures a frequency spectrum. Finally, if the field changes slowly (on the timescale of the measurement time T), we can write

$$j_{\omega_c}(t) \propto T \int d\tau e^{i\omega_c \tau} \langle \mathbf{E}^-(t - \tau/2) \cdot \mathbf{E}^+(t + \tau/2) \rangle.$$

This quantity is called the ‘Wigner transform’ of $\langle \mathbf{E}^-(t) \cdot \mathbf{E}^+(t') \rangle$, a function that combines information in real time and in frequency space.

Chapter 3

Dissipation and decoherence

Introduction

Second aspect of atom + field interactions: “effective dynamics” of system coupled to “large reservoir” whose dynamics is not observed (or not observable). No longer possible to work with pure states. Density matrix formalism or “diffusion process” in Hilbert space (stochastic equation for state vector).

3.1 Master equation for two-level atom

Motivation

Experimentally observed: excited states decay at some rate γ to the ground state. Evolution of population: *rate equation*

$$\begin{aligned}\dot{p}_e &= -\gamma p_e \\ \dot{p}_g &= \gamma p_e\end{aligned}\tag{3.1}$$

Total population conserved. Stationary state: $p_e = 0$.

Problem: not possible to obtain with a Hamiltonian. Standard interaction $V = \frac{1}{2}\hbar\Omega(|g\rangle\langle e| + |e\rangle\langle g|)$ couples population $p_e = \langle e|\rho|e\rangle = \rho_{ee}$ to an off-diagonal matrix element (a “coherence”)

$$\dot{p}_e = -\frac{i}{\hbar}\langle e|[V, \rho]|e\rangle = i\frac{\Omega}{2}(\langle e|\rho|g\rangle - \langle g|\rho|e\rangle) = \Omega \text{Im} \langle g|\rho|e\rangle$$

Challenge: find quantum description for decay processes and combine with coherent dynamics (like Rabi oscillations).

Idea: density matrix ρ for atom+field system. Take the trace over the field and get the *reduced density matrix* ρ_A for the atom:

$$\rho_A = \text{tr}_F \rho.$$

Use second order perturbation theory (“weak interaction”) combined with a separation of “slow” and “fast” timescales: “fast” = fluctuations of the field (short correlation time τ_c); “slow” = dissipative dynamics (decay) of the atom. Recall that $\tau_c \approx \hbar/k_B T \approx 10^{-12}$ s for a thermal field (“fast”) and $1/\gamma \approx 10^{-9}$ s (“slow”) for typical lifetimes. And also $1/\omega_A \approx 10^{-14}$ s for a typical optical period (“very fast”). The resulting equation is “better” than simple perturbation theory — it allows to describe the atomic dynamics on slow timescales, even longer than a decay time.

The calculation we present is due to Wigner and Weis(s?)kopf. Our presentation is inspired by the chapter IV.B of the book “Atom-photon interactions” by Cohen-Tannoudji, Dupont-Roc, and Grynberg.

3.1.1 Density matrix, interaction picture

Schrödinger equation for the atom+field density matrix

$$\frac{d\rho}{dt} = \frac{1}{i\hbar} [H, \rho]$$

with the Hamiltonian

$$H = H_A + H_F + H_{\text{int}}$$

Atom plus laser mode: describe both by H_A .

Interaction picture: unitary transformation

$$\rho(t) = e^{-i(H_A+H_F)t/\hbar} \tilde{\rho}(t) e^{i(H_A+H_F)t/\hbar}$$

gives Schrödinger equation

$$\frac{d\tilde{\rho}}{dt} = \frac{1}{i\hbar} [H_{\text{int}}(t), \tilde{\rho}] \quad (3.2)$$

where the time-dependence of the interaction is that of the free evolution

$$H_{\text{int}}(t) = e^{i(H_A+H_F)t/\hbar} H_{\text{int}} e^{-i(H_A+H_F)t/\hbar},$$

as we saw before. In the following, we drop the tilde to simplify the writing.

Formal solution to (3.2):

$$\rho(t + \Delta t) = \rho(t) + \frac{1}{i\hbar} \int_t^{t+\Delta t} dt_1 [H_{\text{int}}(t_1), \rho(t_1)]$$

Iteration to second order:

$$\begin{aligned} \rho(t + \Delta t) = & \rho(t) + \frac{1}{i\hbar} \int_t^{t+\Delta t} dt_1 [H_{\text{int}}(t_1), \rho(t)] \\ & - \frac{1}{\hbar^2} \int_t^{t+\Delta t} dt_1 \int_t^{t_1} dt_2 [H_{\text{int}}(t_1), [H_{\text{int}}(t_2), \rho(t_2)]] \quad (3.3) \end{aligned}$$

3.1.2 Reduced atom dynamics

Now take the trace over the field to find $\rho_A(t + \Delta t)$. Specify the initial conditions and the interaction.

- Initial conditions: atom+field state factorizes

$$\rho(t) = \rho_A(t) \otimes \rho_F \quad (3.4)$$

Take arbitrary state $\rho_A(t)$ for the atom and a thermal state for the field. Idea: field is a “big” system and its state changes a little when coupled to the atom.

- Interaction: as usual, electric dipole interaction

$$H_{\text{int}}(t) = -\mathbf{d}(t) \cdot \mathbf{E}(t)$$

The time-dependence is that of the free evolution.

Immediate consequence: the trace over the field of the first order term in (3.3) vanishes:

$$\text{tr}_F (H_{\text{int}}(t)\rho(t)) = -\rho_A \mathbf{d}(t) \cdot \text{tr} (\mathbf{E}(t)\rho_F) = 0$$

because the average electric field vanishes in a thermal state.

Hence, the dynamics will be given by the second order term. Note that the unknown density matrix $\rho(t_2)$ appears under the integral in (3.3). In the spirit of perturbation theory, we replace it by the initial $\rho(t)$. If we took into account corrections, they would be due to the interaction, and

the double integral is already of second order in H_{int} . This approximation makes the evolution of the reduced density matrix only dependent on its value at time t , it is called the “Markov approximation” and one says that the field has an “infinitely short memory time”.

We note that the assumption (3.4), that at time t , the atom+field density matrix factorizes, cannot be true at all times. The atom-field interaction creates correlations, and these are precisely responsible for getting a nonzero result at second order. Thus, we can only approximate the full density matrix by a factorized one. The idea is that the atom-field correlations decay on the fast timescale given by the field correlation time τ_c , and that after a time step Δt , they have died out. The Markov approximation precisely assumes that this decay happens quasi-instantaneously. Note also that when we work at zero temperature, we assume that the photons the atom possibly emits “disappear” in the field reservoir (since its state remains the vacuum state). This means that the master equation cannot describe the re-absorption of spontaneously emitted radiation, as it may happen in front of a mirror or in a cavity. For this, either one must use the correct mode functions for the geometry (the van der Waals shift then appears as a second-order level shift), or one must single out the cavity mode from the field and combine it with the atom into the “system” Hamiltonian.

Field correlations

Now we have to deal with the average of two electric field operators (a first-order coherence function). For example, the first term of the double commutator involves

$$1 : \quad \text{tr}_F (E_i(t_1) E_j(t_2) \rho_F) =: C_{ij}(\tau)$$

This coherence function only depends on the time difference $\tau = t_1 - t_2$ because the field is in a stationary (in particular, thermal) state and its density matrix commutes with the field Hamiltonian. *Proof:* Under the trace, we permute cyclically and find

$$\begin{aligned} & \text{tr}_F \left(e^{iH_F t_1/\hbar} E_i e^{iH_F(t_2-t_1)/\hbar} E_j e^{-iH_F t_2/\hbar} \rho_F \right) \\ &= \text{tr}_F \left(e^{iH_F(t_1-t_2)/\hbar} E_i e^{iH_F(t_2-t_1)/\hbar} E_j \rho_F \right). \end{aligned}$$

This is a general property of the coherence function of stationary states. A field in a stationary state is thus “stationary” in the sense mentioned in the exercises (sounds consistent): its coherence function only depends on time differences. (Note, however, that we deal here with the full field operators and not a normally ordered coherence function.)

In the second term in the double commutator, the fields $\mathbf{E}(t_1)$ and $\mathbf{E}(t_2)$ appear in reverse order, leading to the coherence function $C_{ji}(t_2 - t_1) = C_{ji}(-\tau)$.

The coherence function $C_{ij}(\tau)$ of the electromagnetic field is related to the blackbody spectrum, as discussed in the exercises. In particular, in free space, it is proportional to the Kronecker δ_{ij} , and we can write $C_{ij}(\tau) = \delta_{ij}C(\tau)$. We also note that $C(-\tau) = [C(\tau)]^*$.

The four terms in the expansion of the double commutator thus give (the numbers on the left hand side simply enumerate these terms)

$$\begin{aligned} 1 : & \quad \mathbf{d}(t_1) \cdot \mathbf{d}(t_2) \rho_A C(\tau) \\ 2 : & \quad -d_i(t_2) \rho_A d_i(t_1) C(\tau) \\ 3 : & \quad -d_i(t_1) \rho_A d_i(t_2) C(-\tau) \\ 4 : & \quad \rho_A \mathbf{d}(t_2) \cdot \mathbf{d}(t_1) C(-\tau) \end{aligned}$$

Summation over the index i is understood. The terms 1, 4 are hermitean conjugates of each other. Same for the pair 2, 3.

Dipole operator

To proceed, we specify the time-dependence of the dipole operator

$$\mathbf{d}(t) = \mathbf{d}\sigma_+(t) + \mathbf{d}^*\sigma_-(t) = \mathbf{d}\sigma_+e^{i\omega_A t} + \mathbf{d}^*\sigma_-e^{-i\omega_A t}$$

where $\mathbf{d} = \langle e|\hat{\mathbf{d}}|g\rangle$ is the (complex) vector of matrix elements for the dipole operator. Noting that $\sigma_+^2 = \sigma_-^2 = 0$, we find for the product of dipole operators

$$\mathbf{d}(t_1) \cdot \mathbf{d}(t_2) = |\mathbf{d}|^2 \sigma_+\sigma_-e^{i\omega_A \tau} + |\mathbf{d}|^2 \sigma_-\sigma_+e^{-i\omega_A \tau}$$

The term 1 is thus equal to

$$1 : \quad C(\tau) |\mathbf{d}|^2 \left(\sigma_+\sigma_-e^{i\omega_A \tau} + \sigma_-\sigma_+e^{-i\omega_A \tau} \right) \rho_A \quad (3.5)$$

Note that it only depends on the time difference τ .

Short and long timescales

This suggests that the other time integral can be done, giving Δt . More precisely, using the new variables t_1 and τ , we can write

$$\int_t^{t+\Delta t} dt_1 \int_t^{t_1} dt_2 = \int_t^{t+\Delta t} dt_1 \int_0^{t_1-t} d\tau = \int_0^{\Delta t} d\tau \int_{t+\tau}^{t+\Delta t} dt_1$$

In the t_1, t_2 -plane, the integration domain is a triangle below the diagonal. In the last step, we have parametrized this triangle by lines of fixed τ , i.e. parallel to the diagonal, starting at $t_1 = t + \tau$ and ending at $t_1 = t + \Delta t$, as shown in figure 3.1.

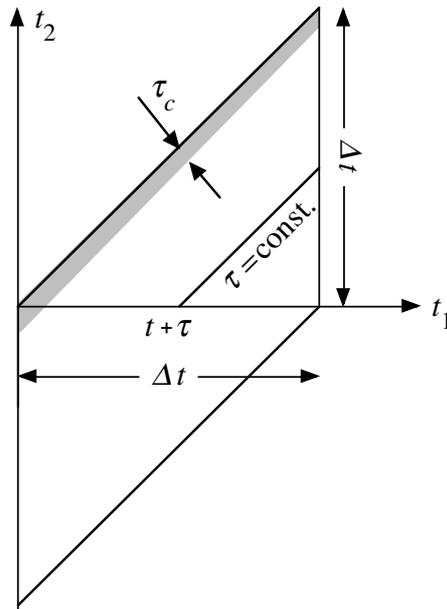


Figure 3.1: Domain of integration in the t_1, t_2 -plane. Due to the short coherence time τ_c of the field, only the gray area just below the diagonal does contribute to the integral.

We now take advantage of the short correlation time of the field: as a function of τ , the coherence function $C(\tau)$ decreases on a scale given by the coherence time τ_c . Since we are interested in the “slow” dynamics of the atom, the time interval Δt can be chosen large compared to τ_c . In this limit, we can also replace the integration domain by the parallelogram shown in

fig. 3.1. This means that the integration limits for the t_1 integral change to

$$\int_0^{\Delta t} d\tau \int_{t+\tau}^{t+\Delta t} dt_1 \approx \int_0^{\Delta t} d\tau \int_t^{t+\Delta t} dt_1.$$

Since the term 1 (3.5) does not depend on t_1 , the integral over t_1 yields Δt . Term 4 (its hermitean conjugate) behaves in the same way. Such a simplification does not hold for the terms 2 and 3. Here, we have to invoke a different argument. Spelling out the dependence on τ and t_1 , we have

$$\begin{aligned} 2 : & \quad -C(\tau) \left(d_i \sigma_+ e^{i\omega_A(t_1-\tau)} + d_i^* \sigma_- e^{-i\omega_A(t_1-\tau)} \right) \\ & \quad \times \rho_A \left(d_i \sigma_+ e^{i\omega_A t_1} + d_i^* \sigma_- e^{-i\omega_A t_1} \right) \end{aligned}$$

The product gives either terms independent of t_1 or involving $e^{\pm 2i\omega_A t_1}$. If $\omega_A \Delta t \gg 1$, these oscillating terms average out when integrating over t_1 . Note that if the optical period is shorter than the correlation time τ_c of the field, this approximation is even less severe than the condition $\Delta t \gg \tau_c$ we already used. It is thus consistent to make this “secular approximation” as it is called; note its similarities to the rotating wave approximation.

Doing so, the term 2 reduces to something independent of t_1 and proportional to the squared dipole matrix element:

$$2 : \quad -C(\tau) |\mathbf{d}|^2 \left(e^{i\omega_A \tau} \sigma_- \rho_A \sigma_+ + e^{-i\omega_A \tau} \sigma_+ \rho_A \sigma_- \right). \quad (3.6)$$

The t_1 integral again gives Δt .

Field spectrum

We now have to treat the integral over τ that is still left. It contains the following two integrals

$$\int_0^{\Delta t} d\tau C(\tau) e^{\pm i\omega_A \tau}.$$

(The terms with $C(-\tau)$ can be expressed in terms of the complex conjugate.) Since we work on “long timescales” $\Delta t \gg \tau_c$, the correlation function is already zero at the upper limit, and we can replace this limit by $+\infty$ without much error. We thus get a “one-sided” Fourier integral. We write the correlation function in terms of its (normal) Fourier expansion,

$$C(\tau) = \int \frac{d\omega}{2\pi} S(\omega) e^{-i\omega\tau}$$

where $S(\omega)$ is essentially the blackbody spectrum. The relation $C(-\tau) = [C(\tau)]^*$ entails that $S(\omega)$ is real. With the following formula for the τ integral

$$\int_0^\infty d\tau e^{i(\omega_A - \omega)\tau} = \pi\delta(\omega_A - \omega) - i\mathcal{P}\frac{1}{\omega - \omega_A}$$

(\mathcal{P} means the principal value), we find

$$\Delta t \gg \tau_c : \quad \int_0^{\Delta t} d\tau C(\tau) e^{\pm i\omega_A \tau} = \frac{1}{2}S(\pm\omega_A) - i\mathcal{P}\int \frac{d\omega}{2\pi} \frac{S(\omega)}{\omega \mp \omega_A}. \quad (3.7)$$

To simplify the notation, we shall write the second term as $-iR(\pm\omega_A)$.

The first and second terms are thus given by

$$\begin{aligned} 1 : \quad & \frac{1}{2}|\mathbf{d}|^2 (S(\omega_A)\sigma_+\sigma_- + S(-\omega_A)\sigma_-\sigma_+) \rho_A \\ & - i|\mathbf{d}|^2 (R(\omega_A)\sigma_+\sigma_- + R(-\omega_A)\sigma_-\sigma_+) \rho_A \end{aligned} \quad (3.8)$$

$$\begin{aligned} 2 : \quad & -|\mathbf{d}|^2 \left[\left(\frac{1}{2}S(\omega_A) - iR(\omega_A) \right) \sigma_-\rho_A\sigma_+ \right. \\ & \left. + \left(\frac{1}{2}S(-\omega_A) - iR(-\omega_A) \right) \sigma_+\rho_A\sigma_- \right] \end{aligned} \quad (3.9)$$

When adding term 3 (the complex conjugate of term 2), the imaginary parts involving $R(\pm\omega_A)$ drop out because the operators $\sigma_-\rho_A\sigma_+$ and $\sigma_+\rho_A\sigma_-$ are hermitean. But $R(\pm\omega_A)$ survives in terms 1 and 4.

3.1.3 The master equation

Coarse grained derivative

We now collect our results. Recalling that the double integral reduces to something proportional to Δt , we can make a difference quotient appear on the left hand side. This gives the following “master equation”

$$\frac{\Delta\rho_A}{\Delta t} = \frac{1}{i\hbar} [H^{(2)}, \rho_A] + \mathcal{L}[\rho_A] \quad (3.10)$$

The terms on the right hand side are discussed in a minute. This difference quotient can be written as a derivative averaged over the “slow” timescale

$$\frac{\Delta\rho_A}{\Delta t} = \frac{1}{\Delta t} \int_t^{t+\Delta t} dt_1 \frac{d\rho_A(t_1)}{dt}$$

Such an average is sometimes called “coarse grained” (*grobkörnig*) because one is not interested in the rapid fluctuations that occur on timescales faster than Δt .

In the following, we focus on long timescales and use the notation $\Delta\rho/\Delta t \mapsto d\rho/dt$, not making the difference between the ordinary derivative and its coarse grained average.

Level shifts

On the right hand side of the master equation (3.10), we have a “Hamiltonian-like” term that is given by

$$\begin{aligned} H^{(2)} &= -\frac{|\mathbf{d}|^2}{\hbar} (R(\omega_A)\sigma_+\sigma_- + R(-\omega_A)\sigma_-\sigma_+) \\ &= -\frac{|\mathbf{d}|^2}{\hbar} (R(\omega_A)|e\rangle\langle e| + R(-\omega_A)|g\rangle\langle g|) \end{aligned} \quad (3.11)$$

This Hamiltonian describes the energy shifts due to the coupling with the field. It is very similar to the Lamb shift we saw in chapter 3. Indeed, for the ground state, we get

$$\langle g|H^{(2)}|g\rangle = -\frac{|\mathbf{d}|^2}{\hbar} R(-\omega_A) = -\frac{|\mathbf{d}|^2}{\hbar} \mathcal{P} \int \frac{d\omega}{2\pi} \frac{S(\omega)}{\omega + \omega_A}.$$

Let us focus on the field at zero temperature, as in chapter 3. The spectrum is then given by

$$\begin{aligned} S_{ij}(\omega) &= \int d\tau e^{i\omega\tau} \langle E_i(\tau) E_j(0) \rangle \\ &= \sum_{\mathbf{k}\mu} E_k^2 \varepsilon_{\mathbf{k}\mu}^i \varepsilon_{\mathbf{k}\mu}^{j*} \underbrace{\langle a_{\mathbf{k}\mu} a_{\mathbf{k}\mu}^\dagger \rangle}_{=1} \int d\tau e^{i(\omega - \omega_k)\tau} \\ &= 2\pi \sum_{\mathbf{k}\mu} \delta(\omega - \omega_k) E_k^2 \varepsilon_{\mathbf{k}\mu}^i \varepsilon_{\mathbf{k}\mu}^{j*} \end{aligned} \quad (3.12)$$

Putting this into the ground level shift, we recover the formula (??) for the Lamb shift:

$$\langle g|H^{(2)}|g\rangle = -\sum_{\mathbf{k}\mu} \frac{E_k^2 |\mathbf{d} \cdot \boldsymbol{\varepsilon}|^2}{\hbar(\omega_k + \omega_A)}. \quad (3.13)$$

As an exercise, you can work out the corresponding expression for the excited state shift. These shifts are usually ignored by incorporating them

into the atomic Hamiltonian, $H_A \mapsto H_A + H^{(2)}$. In fact, the coupling to the electromagnetic field is always there and shifts the energy levels. The observed atomic transition frequency thus already contains them, and the “bare” (not shifted) frequency ω_A that we put at the beginning is not observable as such. This procedure is called “renormalization” and is the second trick to handle the infinities that occur in quantum electrodynamics.

For quantum optics applications, the level shift is typically taken into account when the atom is placed close to a mirror or other macroscopic objects. The modification of the Lamb shift (the van der Waals potential) then enters the master equation as an additional potential and shifts the atomic transition frequency in a position-dependent way. (One also finds that the atomic decay rate, discussed below, is changed in front of a mirror.)

Decay rates

The last term, $\mathcal{L}[\rho_A]$, of the master equation (3.10) contains the “superoperator” \mathcal{L} or “Liouvillian” that is a linear mapping of the density matrix. It cannot be written in terms of a commutator. Collecting the terms of our derivation, it has the following form

$$\begin{aligned} \mathcal{L}[\rho_A] = & -\frac{\gamma_e}{2} \{\sigma_+\sigma_-, \rho_A\} + \gamma_e \sigma_- \rho_A \sigma_+ \\ & -\frac{\gamma_g}{2} \{\sigma_-\sigma_+, \rho_A\} + \gamma_g \sigma_+ \rho_A \sigma_- \end{aligned} \quad (3.14)$$

where $\{A, B\} = AB + BA$ is the anti-commutator and where the rates $\gamma_{e,g}$ are given by

$$\gamma_g^e = \frac{|\mathbf{d}|^2}{\hbar^2} S(\pm\omega_A) \quad (3.15)$$

Note that the Liouvillian (3.14) also conserves the trace of the density matrix (as it must).

Rate equations

The Liouvillian superoperator \mathcal{L} describes decay processes. To see this, let us work out the equations of motion for the populations $p_{e,g}$, i.e., the diagonal elements of the density matrix. Using again $\sigma_+\sigma_- = |e\rangle\langle e|$ and

$\sigma_- \sigma_+ = |g\rangle\langle g|$, we get

$$\begin{aligned}\dot{p}_e &= -\gamma_e p_e + \gamma_g p_g \\ \dot{p}_g &= -\gamma_g p_g + \gamma_e p_e\end{aligned}\tag{3.16}$$

We see here explicitly that the total population is conserved. We recover the rate equations (3.1) when $\gamma = \gamma_e$ and $\gamma_g = 0$, as is the case for a field at zero temperature. In this case, the rate γ_e is the spontaneous decay rate of the excited state. This can be seen using the field spectrum (3.12):

$$\gamma_e = \frac{2\pi}{\hbar^2} \sum_{\mathbf{k}\mu} \delta(\omega_A - \omega_k) E_k^2 |\mathbf{d} \cdot \boldsymbol{\varepsilon}|^2 = \frac{2\pi}{\hbar} \sum_{\mathbf{k}\mu} \delta(E_{e0} - E_{g1}) |\langle g; 1_{\mathbf{k}\mu} | \mathbf{d} \cdot \mathbf{E} | e; 0 \rangle|^2$$

The last expression is Fermi's Golden Rule for the transition rate between the state $|e; 0\rangle$ into the continuum of states $|g; 1_{\mathbf{k}\mu}\rangle$ due to the electric dipole coupling. At nonzero temperature, also the ground state can “decay” via the absorption of thermally excited photons.

The stationary state of the rate equations (3.16) is attained at large times and reads

$$\begin{aligned}p_e(t \rightarrow \infty) &= \frac{\gamma_g}{\gamma_g + \gamma_e} \\ p_g(t \rightarrow \infty) &= \frac{\gamma_e}{\gamma_g + \gamma_e}\end{aligned}\tag{3.17}$$

For zero temperature, we find $p_g(t \rightarrow \infty) = 1$. For finite temperature, we can expect that we recover the Boltzmann factor:

$$\frac{p_e(t \rightarrow \infty)}{p_g(t \rightarrow \infty)} = \frac{\gamma_g}{\gamma_e} = e^{-\hbar\omega_A/k_B T}$$

It is a simple exercise to show that the ground state excitation rate is proportional to the mean photon number at the transition frequency (stimulated absorption), $\gamma_g \propto \bar{n}(\omega_A)$, while for the excited state decay, stimulated emission gives $\gamma_e \propto 1 + \bar{n}(\omega_A)$. Using the mean photon number $\bar{n}(\omega_A) = (e^{\hbar\omega_A/k_B T} - 1)^{-1}$, we indeed get the Boltzmann factor:

$$\frac{\bar{n}(\omega_A)}{1 + \bar{n}(\omega_A)} = \frac{1}{e^{\hbar\omega_A/k_B T} - 1} \frac{e^{\hbar\omega_A/k_B T} - 1}{e^{\hbar\omega_A/k_B T} - 1 + 1} = e^{-\hbar\omega_A/k_B T}.$$

Decoherence

Finally, the real benefit of our master equation is the equation of motion for the off-diagonal elements (the coherences) of the density matrix.

From (3.10), we get for example

$$\frac{d\rho_{eg}}{dt} = -\frac{\gamma_e + \gamma_g}{2}\rho_{eg} \quad (3.18)$$

and its complex conjugate. We observe that the coherences decay due to the coupling to the field. This process is often called “decoherence” — quantum superpositions (with nonzero off-diagonal density matrix elements) turn into classical probabilistic alternatives (described by diagonal density matrices) due to the interaction with some “environment”. The decoherence rate is sometimes written $1/T_2$ in distinction to the rate $\gamma_e = 1/T_1$ for the populations. Note that there exist couplings where one gets a decoherence rate that is larger than the value $(\gamma_e + \gamma_g)/2$ obtained here. This value is a lower limit, except special cases (a “squeezed reservoir”, for example). In the important case of zero temperature (more precisely, $\hbar\omega_A \gg k_B T$), the “optical coherence” ρ_{eg} decays with $\gamma_e/2$, hence at half the rate of the excited state population.

3.1.4 Coherent plus dissipative dynamics

For completeness, we give here the master equation for a two-level atom driven by a monochromatic laser field with Rabi frequency Ω and coupled to the electromagnetic vacuum field.

We use of course the frame rotating at the laser frequency ω . It is simple to show that the corresponding unitary transformation leaves the master equation unchanged: only products $\sigma_+\sigma_-$ appear, and the phase factors $\sigma_\pm \mapsto \sigma_\pm e^{\pm i\omega t}$ cancel. The atomic Hamiltonian is this given by the expression (??) of chapter ??

$$H_A = -\frac{\hbar\Delta}{2}\sigma_3 + \frac{\hbar\Omega}{2}(\sigma_+ + \sigma_-) \quad (3.19)$$

The detuning $\Delta = \omega - \omega_A$ contains, of course, the renormalized transition frequency. The Rabi frequency Ω can be chosen real.

The full master equation, written in terms of the standard representation (not the interaction representation), contains also a commutator

$$\frac{d\rho_A}{dt} = -\frac{i}{\hbar}[H_A, \rho_A] + \mathcal{L}[\rho_A] \quad (3.20)$$

The elements of the density matrix thus evolve according to

$$\frac{d\rho_{ee}}{dt} = -i\frac{\Omega}{2}(\rho_{ge} - \rho_{eg}) - \gamma_e\rho_{ee} \quad (3.21)$$

$$\frac{d\rho_{eg}}{dt} = i\Delta\rho_{eg} - i\frac{\Omega}{2}(\rho_{gg} - \rho_{ee}) - \frac{\gamma_e}{2}\rho_{eg} \quad (3.22)$$

$$\frac{d\rho_{gg}}{dt} = i\frac{\Omega}{2}(\rho_{ge} - \rho_{eg}) + \gamma_e\rho_{ee} \quad (3.23)$$

We thus learn that, starting from the ground state, the laser field first creates the coherence ρ_{eg} between the ground and excited states — and this only if there is a population difference. The coherence is then turned into an excited state population — if it contains a nonzero imaginary part. These equations give the basic description for many physical phenomena: absorption and emission of light, lasing, resonance fluorescence etc.

Approximations

Let us collect the approximations that we had to make.

1. weak coupling between atom and field. We used second order perturbation theory.
2. effective evolution of the atom occurs at long timescales. We found $1/\gamma_e \approx 10^{-9} \text{ s} \gg \tau_c \approx 10^{-12} \text{ s}$. This allows to choose $\tau_c \ll \Delta t \ll 1/\gamma_e$. Note that the field correlation time τ_c is only “short” when the field has a “broad” frequency spectrum $S(\omega)$ (width much larger than γ_e).
3. the period of the emitted radiation is much shorter than $1/\gamma_e$. O.k. for atoms where $\omega_A \gg \gamma_e$.
4. the field state is unaffected by the presence of the atom. Radiated photons escape from the atom and do not interact a second time. The field is a large “reservoir”.

Exercises

There approximately hundreds of exercises you could do once arrived at the master equation. About the first half proposes variations of its derivation, the other half deals with the “optical Bloch equations” (3.21–3.23).

Other master equations. Confirm that the Liouvillian superoperator is unchanged when (i) the field contains one mode in a coherent state (usual description for a laser field). **Hint.** Use the unitary displacement operator to transform between a field in vacuum and with a single mode in a coherent state; (ii) when one comes back from the interaction representation to the standard representation. **Hint.** You have to use the approximation of a very short coherence time τ_c for the field; (iii) when the rotating frame is used for the interaction with the laser field.

Fill the other gaps of the derivation presented in these notes: dependence of the decay rates $\gamma_{e,g}$ on temperature etc.

Derive a master equation for “harmonic oscillator atom”, with the replacements $\sigma_+ \mapsto a^\dagger$ and $\sigma_- \mapsto a$. Consider the limits $\omega_A \tau_c \gg 1$ (decay of a cavity field) and $\omega_A \tau_c \ll 1$ (decoherence of a macroscopic object). In the latter case, you cannot make the r.w.a.

Derive a master equation for a coupling of the form $H_{\text{int}} = \hbar \dot{\varphi} \sigma_3$. The quantity $\dot{\varphi}$ describes frequency fluctuations of the atom (why?). Assuming as before that the correlation function $\langle \dot{\varphi}(t+\tau) \dot{\varphi}(t) \rangle$ decays on a very short timescale, show that the resulting Liouvillian only contains decoherence terms, but no decay of populations.

Optical Bloch equations. Confirm the Bloch equations (3.21–3.23). Derive the equation of motion for the Bloch vector at $T \geq 0$.

Compute the polarizability of two-level atom. Definition: average dipole moment per exciting field amplitude, taking only the positive frequency part $\propto e^{-i\omega t}$. **Hint.** Use the stationary solution to the optical Bloch equation. **Observe** that the two-level atom responds in a nonlinear way to the field. This is called “saturation”.

Solve the master equation for Rabi oscillations: focus on the resonant case and consider the limits $\Omega \gg \gamma_e$ and $\Omega \ll \gamma_e$.

Compute the change of entropy of the atomic system, starting from an initial pure state into final thermal atomic state. Use the “von Neumann” definition of the entropy

$$S(\rho) = \text{tr}(\rho \log \rho)$$

where the log is defined in terms of its series expansion, and the trace is most easily evaluated by diagonalizing the density matrix ρ . Special cases:

excited state decay at zero temperature; superposition $(|g\rangle + |e\rangle)/\sqrt{2}$ at infinite temperature. Speculate about the exchange of heat and information between system and reservoir.