Chapter 5

Elementary laser theory

(2–3 lectures)

5.1 Introduction

These notes are taken from a previous course and give more details on the so-called 'semiclassical laser theory' as well. We discuss (i) how the polarization in a medium enters the Maxwell equations for the cavity mode and (ii) how it is connected to the density matrix of an ensemble of two-level atoms.

5.1.1 Qualitative description

What are the typical components of a laser[1]? Without going into details, we can identify two of them:

- some matter that amplifies light ("active medium");
- some device that traps the light around the space filled with the medium ("cavity")

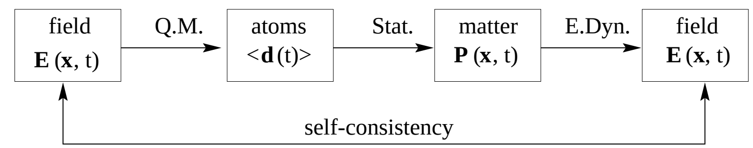
In order to obtain an amplifying medium, one has to "pump" energy into it. The gain medium is thus a converter between the pump energy and the light emission. Quite often, the conversion efficiency is low, with values in the range 10–50% being considered "large".

[1]Acronym of "Light Amplifier by Stimulated Emission of Radiation"

The feedback mechanism is needed because the light would otherwise escape from the medium. An optical cavity like a Fabry-Pérot resonator (two mirrors) does this job because the light can travel back and forth between the mirrors a large number of times.

5.1.2 Idea of the theory

Sargent and Scully (1972) propose the following diagram that relates the different theories needed to describe a laser.



The electromagnetic field drives microscopic dipoles in the laser medium. This was the topic of the previous term. A statistical description gives, implicit in the density matrix approach we followed, links the dipoles to the macroscopic polarization of the medium [see Eq. (5.7) below]. The polarization enters the Maxwell equations for the electromagnetic field as a source and generates the field. In the end, a self-consistent description is required: the fields at the left and right end should coincide. The condition of self-consistency allows to derive the following important quantities:

- the laser threshold,
- the laser intensity in steady state,
- the laser frequency.

This can be achieve even when one treats the field treated classically. For example, a "classical" or "coherent" field appeared via the Rabi frequency in the Bloch equations of the previous term. This approach is called "semi-classical laser theory". Note that nowhere in this approach does the word "photon" appear (if one is serious).

When a quantum-mechanical description is adopted, the photon finally comes into play and one may also derive

- the photon number probabability distribution ("photon statistics"),
- the intensity fluctuations and correlations of laser light,
- the phase fluctuations (related to the laser linewidth).

We shall illustrate this quantum theory by a calculation of the photon statistics and the laser linewidth.

We focus in these elementary considerations on a homogeneously distributed medium in the cavity made up from identical two-level systems ("homogeneous broadening"). Please refer to the experimental physics lectures for the discussion of "inhomogeneous" frequency broadening due to, for example, the atomic motion and other features.

5.2 Semiclassical laser theory

5.2.1 Wave equation for the field

We start with a reminder of the electrodynamics in a material with a given polarization. Let us recall that the polarization field enters the following Maxwell equation:

$$\frac{1}{\mu_0}\nabla\times\mathbf{B} = \mathbf{j} + \frac{\partial}{\partial t}\left(\varepsilon_0\mathbf{E} + \mathbf{P}\right)$$

where it gives the "bound" part of the current density. We put in the following the "free" current density $\mathbf{j} = \mathbf{0}$ because we assume that the active material is globally neutral and the light only induces dipoles in it. Combining with the Faraday induction equation, $\nabla\times\mathbf{E} = -\partial_t\mathbf{B}$, one gets the wave equation for the electric field where the polarization enters as a source term:

$$\nabla\times\nabla\times\mathbf{E} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = -\mu_0\frac{\partial^2}{\partial t^2}\mathbf{P}. \tag{5.1}$$

We now make the approximation that in the cavity, a single mode is sufficient to capture the field dynamics. You have seen that one can then write for the field operator

$$\mathbf{E}(\mathbf{x},t) = \mathbf{e}E_1\left[f(\mathbf{x})a(t) + f^*(\mathbf{x})a^\dagger(t)\right] \tag{5.2}$$

where $E_1 = (\hbar\omega_L/2\varepsilon_0 V)^{1/2}$ is the "one-photon field amplitude", $a(t)$ is the annihilation operator for the mode (time-dependent in the Heisenberg picture), and $f(\mathbf{x})$ is the spatial mode function. It solves the homogeneous equation

$$\nabla\times\nabla\times\mathbf{e}f - \frac{\omega_c^2}{c^2}\mathbf{e}f = \mathbf{0} \tag{5.3}$$

as you remember from the field quantization procedure. Here ω_c is one of the (empty) cavity resonance frequencies. For a planar cavity with axis along the z-direction, for example, we have

$$f(\mathbf{x}) = \sqrt{2}\sin(kz) \tag{5.4}$$

with $k = n\pi/L = \omega_c/c$ where n is a positive integer and L the cavity length. This mode function is normalized such that the integral of its square over the cavity volume V gives V. There are also lasers where propagating modes are a suitable description. In the photonics lectures, other cavity modes, including their transverse behaviour (perpendicular to the cavity axis) are introduced. For the semiclassical theory we develop here first, the product $E_1 a(t) = E(t)$ gives the (positive frequency) field amplitude. Its absolute square corresponds to the intensity, with $a^*(t)a(t)$ giving the "photon number" (although this is not required to be an integer in the semiclassical theory).

We now project the wave equation (5.1) onto the field mode $\mathbf{e}f(\mathbf{x})$. The term involving $f^*(\mathbf{x})a^\dagger(t)$ does not contribute when a propagating mode is used. We also make the approximation of slowly varying amplitudes for $E(t)$ that oscillates essentially at the frequency ω_L. The polarization field as well, $\mathbf{P}(\mathbf{x},t) = \mathbf{P}(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\omega_L t} + \mathrm{c.c.}$ This means that the time derivative of $\mathbf{P}(\mathbf{x})$ is much smaller than $\omega_L\mathbf{P}(\mathbf{x})$. (With this approximation and standing wave modes, the term $f^*(\mathbf{x})a(t)$ drops out at this point.) We get

$$\dot{E} = -\mathrm{i}(\omega_L - \omega_c)E - \frac{\kappa}{2}E + \mathrm{i}\frac{\omega_L}{2\varepsilon_0}\int\frac{\mathrm{d}^3x}{V}f^*(\mathbf{x})\mathbf{e}\cdot\mathbf{P}(\mathbf{x}), \tag{5.5}$$

where $\omega_L - \omega_c$ is the frequency detuning with respect to the cavity resonance and V the cavity volume. We have introduced the phenomenological decay rate κ for the energy of the cavity field. The quality factor of the cavity (often known experimentally) is given by $Q = \omega_c/\kappa$. Notice that the spatial integral is the overlap of the polarization field with the cavity mode.

It is easy to see from this equation that the *real* part of the polarization $\mathbf{P}$ determines a frequency shift of the laser (with respect to the cavity frequency), and that its *imaginary* part changes the energy $\propto |E|^2$ of the field. In particular, if $\operatorname{Im}\mathbf{P}$ is negative, the field energy increases (emission). We thus anticipate to find the absorption and emission of the medium in the imaginary part of the polarization.

5.2.2 Dipole moment and polarization

The "active medium" cavity inside the cavity that provides the polarization consists, in many cases, of a large number of atoms or molecules. These atoms are prepared in the excited state by some process that feeds energy into them ("pumping mechanism"), and then wait to release their energy in the form of photons into the cavity field. A two-level approximation for the atoms is a simple way to account for the sharp, nearly monochromatic emission spectrum of the laser. We could have used as well a harmonic oscillator[2], however, this does not reproduce some basic features of the laser like gain saturation. The quantum theory for the atom-light interaction gives us an expression for the "microscopic", average electric dipole $\langle \mathbf{d}(t) \rangle$. The polarization field is then simply the number density of these dipoles

$$\mathbf{P}(\mathbf{x}, t) = N(\mathbf{x}) \langle \mathbf{d}(t) \rangle. \tag{5.6}$$

In general, the density $N(\mathbf{x})$ is position dependent. In fact, also the induced dipole is because it involves the light field at the position $\mathbf{x}$.

The complex, slowly varying polarization field can be connected to the coherences of the density matrix in the rotating frame:

$$\begin{aligned} \mathbf{P}(\mathbf{x}, t) &= N(\mathbf{x}) \left[\langle \mathbf{d}^{(+)}(t) \rangle + \text{c.c.} \right] \\ &= N(\mathbf{x}) \mathbf{d} \left[\rho_{\text{eg}}(t) \mathrm{e}^{-\mathrm{i}\omega_L t} + \text{c.c.} \right] \end{aligned} \tag{5.7}$$

where $\mathbf{d}^{(+)}(t)$ is the positive frequency part of the atomic dipole operator, $\mathbf{d}$ is the fixed vector of dipole matrix elements and $\rho_{\text{eg}}(t)$ is the off-diagonal element ("optical coherence") of the atomic density matrix in the frame rotating at ω_L. A formula like (5.7) assumes that all dipoles in the medium are driven by a similar field and do not interact with each other. This is a first starting point and leaves place for more elaborate theories, of course. We assume in particular that the resonance frequency is the same for all microscopic dipoles. This is not true for atoms in a (thermal) gas where the Doppler effect leads to a distribution of the resonance frequencies ("inhomogeneous broadening").

[2] This is a good model for antennas emitting at radio frequencies.

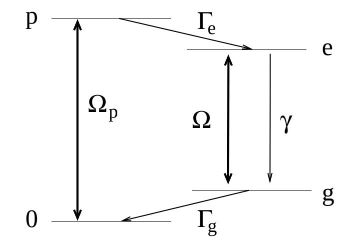


Figure 5.1: Four-level model to describe incoherent population pumping of the upper state e of the lasing transition e $\leftrightarrow$ g.

5.2.3 Dipole moment and atomic inversion

We now have to find a way to determine the dipole moment of the two-level atoms. Recall that its imaginary part is essential for light amplification. We shall see that it depends on the atomic inversion (population difference between upper and lower state). To this end, we use the optical Bloch equations for the atomic density matrix, with some modifications by adding additional energy levels. This model also provides a better understanding of the "pumping" mechanism, beyond some phenomenological rate equations. The modified two-level system is for example a four-level atom with fast relaxation in the two upper and two lower states. A simple model with four states as shown in figure 5.1 is outlined in the exercises. In this limit, the optical Bloch equations can be simplified, and one gets a justification for the often-used rate equations.

The next task is to compute the optical coherence $\rho_{\text{eg}}(t)$ from the optical Bloch equations. Let us write down these equations for the two levels e and g involved in the laser transition. The rate equations for the populations ρ_{ee}

and ρ_{gg} involve a pumping rate $\lambda_e = \Gamma_e \rho_{pp}$ into the upper state (via rapid decay from the pumped state p), the spontaneous decay rate γ and a decay rate Γ_g for the lower state. Including the Rabi frequency $\Omega = -(2/\hbar)\mathbf{d} \cdot \mathbf{E}$ for the laser field, as you have learned in the previous semester, this gives

$$\dot{\rho}_{ee} = \lambda_e - \gamma \rho_{ee} + \mathrm{i}\frac{\Omega}{2}\left(\rho_{eg} - \rho_{ge}\right), \tag{5.8}$$

$$\dot{\rho}_{gg} = \gamma \rho_{ee} - \Gamma_g \rho_{gg} - \mathrm{i}\frac{\Omega}{2}\left(\rho_{eg} - \rho_{ge}\right). \tag{5.9}$$

Notice that the first equation gives an increase of the excited state population when the coherence ρ_{eg} has a positive imaginary part (recall that Ω is actually negative…). This is in agreement with the damping of the field energy in the wave equation derived before.

The last Bloch equation is for the coherence itself. The population decay rates γ and Γ_g lead to a decoherence rate $\Gamma = \frac{1}{2}(\gamma + \Gamma_g)$ as you have seen in the derivation of the Bloch equations. In the frame rotating at the laser frequency ω_L, the laser field detuning is $\Delta = \omega_L - \omega_{eg}$, and we get

$$\dot{\rho}_{eg} = \mathrm{i}\Delta \rho_{eg} - \Gamma \rho_{eg} + \mathrm{i}\frac{\Omega}{2}\left(\rho_{ee} - \rho_{gg}\right). \tag{5.10}$$

From this equation we learn that the optical dipole is created by the population difference (inversion) $\rho_{ee} - \rho_{gg}$. As discussed in the exercises, this equation can be solved approximately in the limit that the decay rate Γ is the largest time constant around (this solution also corresponds to the stationary state):

$$\rho_{eg} = -\frac{\Omega/2}{\Delta + \mathrm{i}\Gamma}\left(\rho_{ee} - \rho_{gg}\right). \tag{5.11}$$

In particular, the imaginary part of the optical coherence is (we assume as usual a real Rabi frequency)

$$\mathrm{Im}\,\rho_{eg} = \frac{\Gamma \Omega/2}{\Delta^2 + \Gamma^2}\left(\rho_{ee} - \rho_{gg}\right).$$

Note that this expression is negative when the two-level system is inverted (upper level population $\rho_{ee} > \rho_g$), using again that actually $\Omega < 0$. This means that the medium amplifies the light via stimulated emission.

5.2.4 Medium polarization and saturation

Solving also the other Bloch equations in the stationary state, we can compute the inversion

$$\rho_{ee} - \rho_{gg} = \lambda_e \left(1/\gamma - 1/\Gamma_g\right) \frac{\Delta^2 + \Gamma^2}{\Delta^2 + \Gamma^2 + (\Gamma/\gamma)\Omega^2/2}. \tag{5.12}$$

The system is inverted when the lifetime $1/\gamma$ of the upper state exceeds the lifetime $1/\Gamma_g$ of the lower state, which is perfectly reasonable.

The end result of the calculation is the following expression for the polarization field. We quote only the amplitude of the positive frequency component and assume that dipole moment and electric field are collinear:

$$\mathbf{P}(\mathbf{x}) = N(\mathbf{x})(D^2/\hbar)\mathbf{e}E(\mathbf{x}) \frac{\lambda_e \left(1/\gamma - 1/\Gamma_g\right)(\Delta - \mathrm{i}\Gamma)}{\Delta^2 + \Gamma^2 + 2(\Gamma D^2/\gamma\hbar^2)|E(\mathbf{x})|^2} \tag{5.13}$$

$$=: \frac{\varepsilon_0 \chi \mathbf{E}(\mathbf{x})}{1 + B|E(\mathbf{x})|^2}. \tag{5.14}$$

In the last line, we have introduced the (linear) susceptibility χ of the laser medium and a coefficient B that takes into account the nonlinear response.

The most important result is that the imaginary part of the polarization is negative (*amplification* of the field) when the two-level system is inverted. The four-level scheme shown in fig. 5.1 is just one possibility to achieve inversion by a suitable pumping scheme. Refer to the experimental physics lectures for other, perhaps more efficient, pumping mechanisms.

The coefficient B in (5.14) describes the *saturation* of the medium: for very large laser intensity $|E|^2$, the induced polarization decreases proportional to $1/|E|$ instead of increasing. The physics behind saturation is characteristic for the two-level system: when the laser field gets extremely strong, the inversion vanishes (see Eq. (5.12)). We have already seen this behaviour when we considered Rabi oscillations with weak damping: the two-level system gets always re-excited by the laser and is finally with equal probability in the upper and lower states. For a harmonic oscillator, there is no saturation since arbitrarily high lying states can be populated.

5.2.5 Laser threshold and steady state

We now want an equation for the intensity $I(t) = |E(t)|^2$ (restoring a slow time-dependence) in the mode $\mathbf{e}f(\mathbf{x})$. To this end, we work out the spatial overlap in Eq. (5.5) with a standing wave mode $f(\mathbf{x}) = \sqrt{2}\sin kz$:

$$\dot{E} = -\mathrm{i}(\omega_L - \omega_c)E - \frac{\kappa}{2}E + \mathrm{i}\frac{\omega_L \chi}{2}E(t)\int \frac{\mathrm{d}z}{L}\frac{2\sin^2(kz)}{1 + 2B|E(t)|^2\sin^2(kz)}$$

Here, L is the cavity length. The difficulty is the sine function in the denominator. In the exercises, you are asked to compute this integral analytically. Here, we adopt an approximate treatment that is also often used in the literature and assume that the saturation is weak. The denominator can then be expanded, and to first order in B, we get

$$\int \frac{\mathrm{d}z}{L}2\sin^2(kz)\left[1 - 2B|E(t)|^2\sin^2(kz)\right] = 1 - \frac{3B}{2}|E(t)|^2$$

As an exercise, you can estimate the dimensionless quantity $B|E(t)|^2$ for typical parameter values. This result is often "resummed" to make the saturation effect more clear:

$$1 - \frac{3B}{2}|E(t)|^2 \approx \frac{1}{1 + \frac{3B}{2}|E(t)|^2}$$

This procedure may seem strange, but reproduces quite well the exact result, as shown in figure 5.2.

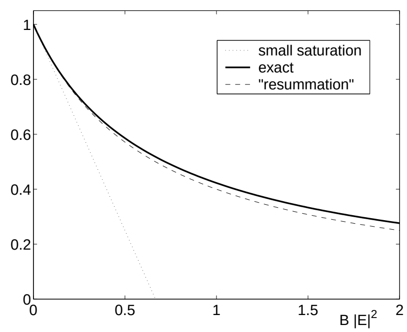


Figure 5.2: Different approximations for the gain saturation factor, integrated over the cavity mode.

The equation of motion for the intensity is now given by (for small gain saturation; check that B is real)

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 2\,\mathrm{Re}\left(E^*\frac{\mathrm{d}E}{\mathrm{d}t}\right) = -\kappa I(t) - \omega_L(\mathrm{Im}\,\chi)I(t)\left[1 - (3B/2)I(t)\right]$$

This form suggests the introduction of an amplification rate ("gain") $G = -\omega_L(\mathrm{Im}\,\chi)$. Using also the conventional notation $\beta = 3GB/2$ (a rate per intensity) for the saturation coefficient, we obtain

$$\frac{\mathrm{d}I}{\mathrm{d}t} = (G - \kappa)I - \beta I^2 \tag{5.15}$$

as the fundamental equation of motion for the laser emission.

Threshold

The *laser threshold* is reached when the gain is sufficiently large to amplify the laser intensity: $G > \kappa$. In this case, stimulated emission overcomes the loss of the field due to cavity imperfections etc. If the gain is too small (or even negative), $G < \kappa$, then the field decays exponentially from its initial value. This remark points to a weakness of the semiclassical approach. It is in fact experimentally known that, once the laser is above threshold (gain exceeds loss), the laser field builds up — although one starts from an apparently empty cavity. What happens is that the spontaneous emission of the laser medium "ignites" the exponential growth of the laser field. But a model for this requires a quantum treatment of the laser, as we shall outline in the next session.

Steady state intensity

Once the laser is above threshold, the field does not grow indefinitely: the saturation of the medium, described by the coefficient β, stops the exponential growth. A steady state is reached at an intensity

$$I_{\mathrm{ss}} = \frac{G - \kappa}{\beta}.$$

This expression is valid not too far above threshold where the expansion of the saturation denominator is still accurate. Note that this increases linearly with the gain which is itself proportional to the pumping power. A typical diagram is shown in figure 5.3 where the steady state intensity is plotted vs. gain. Experimentalists often use instead of G the pump power.

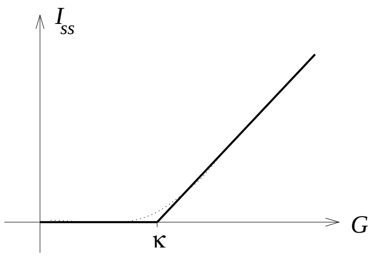


Figure 5.3: Intensity of laser emission vs. gain. The laser threshold is at $G = \kappa$ where κ is the loss rate. Solid line: semiclassical theory (5.16). The dotted line results from the quantum theory.

Phase transition analogy

To summarize, the laser intensity in steady state above and below threshold is given by

$$I_{ss} = \begin{cases} 0 & \text{if } G < \kappa, \\ \dfrac{G - \kappa}{\beta} & \text{if } G > \kappa. \end{cases} \tag{5.16}$$

It has been noted that this behaviour is analogous to a phase transition: as a "control parameter" (gain, temperature) is scanned across a critical value, an "order parameter" (laser intensity, magnetization, long-range order) abruptly changes to a nonzero value. This way of thinking has been explored in depth in the school of H. Haken (Stuttgart), see, e.g., *Light*, vol. 2 (North-Holland, Amsterdam 1985), *Synergetics* (Springer, Berlin 1977), and *The Theory of Coherence, Noise and Photon Statistics of Laser Light*, chap. A3 in *Laser Handbook*, vol. 1, edited by F. T. Arecchi and E. O. Schulz-DuBois (North-Holland, Amsterdam 1972).

Other topics of semiclassical laser theory

These questions are deferred to other lectures:

- time evolution of the laser intensity ("relaxation oscillation") and chaos ("Lorenz model")

- stability analysis of the above threshold solution $I_{ss} = 0$.
- inclusion of "inhomogeneous broadening" (dipoles in motion, distribution of transition frequencies, etc.)
- alternative pumping schemes (incoherent pumping, current injection and laser diodes etc.)
- multi-mode operation, frequency locking of different modes, "Q-switching" etc.

Exercises

Find reasonable values for the parameters gain G, cavity loss κ and saturation coefficient B. An often used quantity is the "saturation intensity" $I_{sat} = 1/B$. It gives the typical order of magnitude for the intra-cavity field.

Compute the laser frequency shift with respect to the cavity resonance ω_c (re-derive carefully the wave equation) and the medium resonance ω_{eg}.

Plot the gain spectrum (as a function of ω_L) for different laser powers $|E|^2$. At low power, you find a lorentzian whose width is given by the "dephasing rate" Γ (the relaxation rate of the optical dipole); at high power, the width increases — this is called "power broadening".

Try to solve the time-dependent nonlinear equation of motion (5.15), if nothing else works, at least numerically. How does one reach the steady state? Do the initial conditions matter?

5.3 Quantum laser theory

In this section, we outline a theory of the laser that starts from a quantum description of the cavity field. We still use for simplicity the single mode approximation — the basic observables are hence the annihilation and creation operators $a, a^\dagger$ for the field mode.

The laser is an *open quantum system* because energy is continuously fed into and removed from the cavity mode. We therefore have to use a density matrix description, as we did in the first part for a two-level atom. What are the "reservoirs" that the field mode is coupled to? First of all, the mode continuum outside the cavity: part of the cavity losses show up here (and permit to observe the laser dynamics). But in general, losses also occur in the material that makes up the cavity: mirrors and optical elements. We do not develop in this semester's course a detailed quantum theory of lossy optical elements (see chapter 4 of the SS 2002 version). Finally, the laser medium is also a reservoir of energy that may flow into the field mode — or not when the medium spontaneously emits photons into other modes.

In this section, we recall the master equation description for linear cavity loss and motivate the corresponding model for the gain medium. We shall derive a rate equation for the probabilities of finding n photons in the laser mode whose stationary solution gives the photon statistics. Finally, a sketch is given of the Schawlow-Townes limit for the laser linewidth.

5.3.1 Cavity damping

The density operator for the cavity field, $\rho(t)$, acts on the Hilbert space for the harmonic oscillator associated with the field mode. Taking the trace, we find the quantum expectation values of the quantities of interest. The average electric field, for example, is given by (we only write the positive frequency part)

$$\langle \mathbf{E}(\mathbf{x},t)\rangle = \mathbf{e} f(\mathbf{x}) E_1 \langle a(t)\rangle = \mathbf{e} f(\mathbf{x}) E_1 \,\mathrm{tr}\,[a\,\rho(t)]$$

The trace can be performed in any basis, using photon number states or coherent states, for example. In the absence of any interaction, the Heisenberg operator a evolves freely at the frequency ω_c of the cavity. (We suppose for simplicity that this coincides with the laser frequency.)

We have treated in the lecture this year the damping of the field in terms of a quantum Langevin equation for the mode operator $a(t)$. Here, we need a formulation for the density matrix (or operator) ρ of the field mode. This is given by, for a loss rate κ of the cavity,

$$\left.\frac{\mathrm{d}\rho}{\mathrm{d}t}\right|_{\mathrm{damp}} = -\frac{\kappa}{2}\left\{a^\dagger a,\, \rho\right\} + \kappa\, a\rho a^\dagger. \tag{5.17}$$

In the exercises, you are asked to check that the rate κ has the same meaning as in the semiclassical theory: it gives the (exponential) decay of the

field's photon number if no other dynamics is present. Note that the rate equation associated with this master equation gives a transition rate between the photon number states $|n\rangle$ and $|n-1\rangle$ that is given by $n\kappa$: it looks as if 'each photon' in the number state $|n\rangle$ has the same probability κ (per unit time) to decay, hence the n-photon state decays n times faster. The final state is the vacuum state with zero photons — this is related to the implicit assumption that the reservoir is at zero temperature. It is a reasonable approximation at optical frequencies and room temperature.

5.3.2 Gain

In the previous semester, we used a model with a driven field mode where a "pump" generates a coherent state. We cannot use this model any longer because the laser medium does not provide, a priori, a fixed phase reference for the field it generates. At least the spontaneous emission of the pumped two-level atoms is "incoherent" (no fixed phase).

We shall see that a suitable model for the cavity gain is given by the master equation

$$\left.\frac{\mathrm{d}\rho}{\mathrm{d}t}\right|_{\mathrm{gain}} = -\frac{G}{2}\left\{a a^\dagger,\, \rho\right\} + G\, a^\dagger\rho a, \tag{5.18}$$

where G is the gain rate coefficient and, up to the exchange of a and $a^\dagger$, no sign changes occur. In the exercises, you check that the average field and photon number indeed increase under this time evolution, as was the case for the semiclassical theory.

How can an equation like (5.18) be motivated? Let us sketch a derivation in the spirit of the Wigner-Weiskopf approach used in the previous semester. We describe the "reservoir" by an ensemble of two-level atoms and use a coupling

$$H_{\mathrm{int}} = \hbar g\left(\sigma_+ a + \sigma_- a^\dagger\right),$$

where the one-photon Rabi frequency is $g = -\mathbf{D}\cdot\mathbf{e}E_1/\hbar$. Note that we have cheated to ignore the position dependence. Using the standard procedure to derive a master equation, we observe that this description is valid on timescales long compared to the correlation time of the atomic ensemble (at most the spontaneous lifetime, if not shorter due to strong pumping or dephasing). We find that the gain coefficient is given by the integral

$$\frac{G}{2} = g^2 \, \mathrm{Re} \int_0^{+\infty} \mathrm{d}\tau \, \langle \sigma_+(t+\tau)\sigma_-(t) \rangle \, \mathrm{e}^{\mathrm{i}(\omega_{\mathrm{eg}} - \omega_{\mathrm{c}})\tau}$$

where the time-correlation function of the dipole operator appears, with the average being taken in the state of the medium. It seems reasonable to take the stationary state, the correlation function is then independent of t. We encountered this integral before — it gives the spectrum of the fluorescence emitted by a two-level atom (at the cavity frequency), that we computed in the section on resonance fluorescence.

Note also this is a very reasonable result — the cavity field is changed because the medium emits photons into it. In the following, we shall continue to use the symbol G for the gain coefficient without writing down explicitly the emission spectrum.

What about gain saturation? It is included in this theory if we allow the fluorescence spectrum to depend on the instantaneous intensity of the cavity mode. The gain thus depends on the photon number, $G = G(n)$. We have seen this dependence when discussing the induced dipole moment of a two-level atom where saturation comes in at large Rabi frequencies. Similar to the semiclassical gain, we can write

$$G(a^\dagger a) = \frac{G_0}{1 + B a^\dagger a}$$

where G_0 is the "linear gain" and $1/B$ a "saturation photon number". Of course, this description is not completely rigorous because our calculation of saturation was never careful enough to allow the Rabi frequency to be a non-commuting operator. We have to fix the precise formula for gain saturation by self-consistency.

Finally, let us mention that the master equation also gives absorption terms for the cavity field by the two-level medium. We can either neglect them, assuming that the two-level medium is essentially in the excited state (absorption is then impossible), or combine them with the cavity damping rate κ.

Summarizing, we get the following master equation including damping and gain:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\mathrm{i}\omega_{\mathrm{c}} \left[a^\dagger a, \, \rho \right] - \frac{\kappa}{2} \left\{ a^\dagger a, \, \rho \right\} + \kappa \, a \rho a^\dagger$$

$$- \frac{G(a^\dagger a)}{2} \left\{ a a^\dagger, \, \rho \right\} + G(a^\dagger a) \, a^\dagger \rho a. \tag{5.19}$$

A more careful derivation, taking into account many atoms with a spatial distribution, gives essentially the same equation, with some numerical factors that change the coefficients G_0 and B. These are in any case only model parameters that have to be chosen in accordance with experiments, so we can continue here.

5.3.3 Photon statistics

To start our analysis, let us compute the rate equations for the populations $p_n \equiv \rho_{nn}$ of finding n photons in the cavity mode. Taking the expectation value in the state $|n\rangle$ of the master equation (5.19), we get

$$\frac{\mathrm{d}p_n}{\mathrm{d}t} = -n\kappa\, p_n + (n+1)\kappa\, p_{n+1} - (n+1)G(n)\, p_n + nG(n)\, p_{n-1} \quad \text{(nearly correct)} \tag{5.20}$$

From the terms with a negative sign, we see that transitions leave the state $|n\rangle$ with rates $n\kappa$ and $(n+1)G(n)$. Looking at the rate equation for the state $|n-1\rangle$, we see that population from state $|n\rangle$ arrives at a rate $n\kappa$. We have thus identified a first process: the cavity field loses one photon at the rate $n\kappa$. This is the expected loss process. But there is also a transition from $|n\rangle$ to $|n+1\rangle$, occurring at a rate $(n+1)G(n)$. This is both spontaneous ("+1") and stimulated emission ("n") from the laser medium. Note that the present theory requires G to be positive (inverted medium) because transition rates are positive. The dependence of $G(n)$ on the photon number again models the gain saturation, as was the case in the semiclassical theory.

The transitions we have found are summarized in figure 5.4. We can

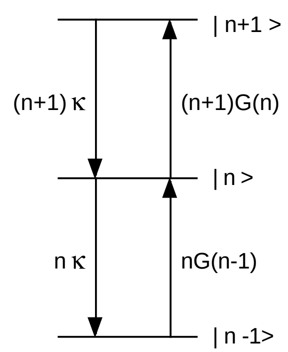


Figure 5.4: Transitions between photon number states.

now determine the stationary state of the laser. The probabilities p_n and p_{n+1}, say, then do not change with time, and therefore the probability current for the loss process $|n+1\rangle \to |n\rangle$ must be equal to the current for the emission process $|n\rangle \to |n+1\rangle$:

$$(n+1)\kappa p_{n+1} = (n+1)G(n)p_n$$

For the pair of levels $|n\rangle$ and $|n-1\rangle$, we get $n\kappa p_n = nG(n)p_{n-1}$ from the rate equation (5.20). But since these relations must hold for every n, we can suspect an error in the gain saturation. The correct relation is

$$n\kappa p_n = nG(n-1)p_{n-1}$$

We can check for the special case $|0\rangle \leftrightarrow |1\rangle$ that there is no saturation effect for the spontaneous emission rate into the empty cavity, which seems perfectly reasonable.[3] The correct rate equation for the photon number distribution is thus

$$\frac{\mathrm{d}p_n}{\mathrm{d}t} = -n\kappa\, p_n + (n+1)\kappa\, p_{n+1} - (n+1)G(n)\, p_n + nG(n-1)\, p_{n-1} \quad \text{(correct)}. \tag{5.21}$$

We observe that the dynamic equilibrium between loss and gain processes gives a recurrence relation for the photon number probabilities. It is easily solved to give

$$p_{n+1} = \frac{G(n)}{\kappa}p_n \Rightarrow p_n = \mathcal{N}\prod_{m=0}^{n}\frac{G(m)}{\kappa} = \mathcal{N}\left(\frac{G_0}{\kappa}\right)^n \prod_{m=0}^{n}\frac{1}{1+Bm}, \tag{5.22}$$

where $\mathcal{N}$ is a normalization constant. Below threshold, $G_0 < \kappa$, each of the ratios $G(n)/\kappa$ is smaller than unity, and the most probable state is the vacuum — perfectly reasonable because the laser intensity is damped away. Above threshold and for weak saturation, $G(n)/\kappa \approx G_0/\kappa > 1$, and photon numbers larger than zero are favoured. The maximum of the distribution is reached at a photon number $n_{\max}$ where $G(n_{\max})/\kappa = 1$. This equation can be solved to give

$$n_{\max} = \frac{G_0 - \kappa}{\kappa B}$$

which looks very similar to the steady state intensity of the semiclassical theory.

[3] There are high Q cavity experiments where the field of a single photon suffices to saturate an atomic transition.

The photon statistics (5.22) is plotted in figure 5.5 for a laser below and above threshold. Note that below threshold, we do not have a thermal state (the probability is not an exponential $\propto \mathrm{e}^{-\beta n\hbar\omega_c}$), and that above threshold, the width of the number distribution is larger than for a coherent state with the same most probable photon number.

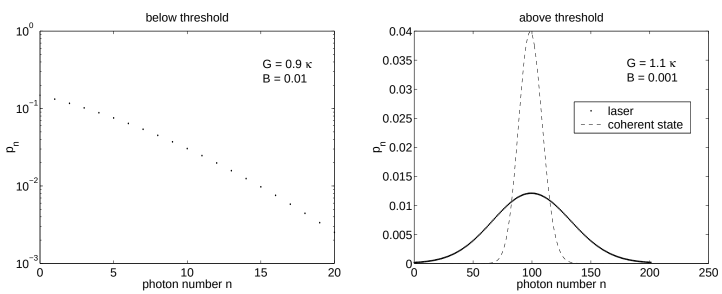


Figure 5.5: Photon statistics of a laser in steady state. Left: below threshold $G \equiv G_0 < \kappa$, right: above threshold.

As an exercise, you can use the following representation of the product in (5.22)

$$\prod_{m=0}^{n} \frac{1}{1+Bm} = B^{-n-1}\frac{\Gamma(1/B)}{\Gamma(1/B+n+1)}$$

where $\Gamma(\cdot)$ is now the gamma function. Using the Stirling formula for large values of n and $1/B$, show that p_n has the form of a truncated gaussian distribution and compute its width. You will find that the width approaches that of a coherent state when the laser is operating far above threshold. (This is difficult to achieve in practice, however.)

5.3.4 Laser linewidth

Up to now, we have assumed that the laser mode oscillates at the frequency ω_L. This is not strictly true. Its frequency spectrum has a finite width. In particular, one observes experimentally that the linewidth gets narrower far above threshold. The corresponding limit of the laser linewidth has been derived in the 1960/70's by Schawlow and Townes. We sketch here a derivation of this formula.

Idea: phase diffusion

The main idea is that the laser field is essentially subject to phase fluctuation, but not to intensity fluctuations. In a semiclassical description, we thus have an electric field amplitude for the laser mode, $E(t) = \sqrt{I}\,\mathrm{e}^{\mathrm{i}\phi(t)}$, where only the phase is fluctuating. We shall see below that the fluctuations of the laser phase are "diffusive" – the phase makes a "random walk". In terms of a distribution function $P(\phi, t)$, this behaviour is described by a diffusion equation,

$$\frac{\partial}{\partial t}P = D\frac{\partial^2}{\partial\phi^2}P \qquad (5.23)$$

where D is called "phase diffusion coefficient". You may remember this equation from heat conduction. Its solution, for an initial state with a fixed phase ϕ_0, is given by a gaussian distribution

$$P(\phi, t|\phi_0) = \frac{1}{\sqrt{4\pi Dt}}\mathrm{e}^{-(\phi-\phi_0)^2/4Dt}, \qquad t > 0. \qquad (5.24)$$

The width of the gaussian is $2Dt$ and increases with time. For very large times, the distribution is completely flat. (This solution neglects the fact that the phase is only defined in the interval $[0, 2\pi]$. See the exercises for this case.) For $t \to 0$, one recovers a δ-function centered at ϕ_0.

With this result, we can compute the temporal correlation function of the laser field,

$$\langle E^*(t+\tau)E(t)\rangle,$$

which gives us the spectrum by a Fourier transform with respect to τ. We can assume that $E(t)$ has a phase $-\omega_L t + \phi_0$ and write $-\omega_L(t+\tau) + \phi(\tau)$ for the phase of $E(t+\tau)$. Taking the average over $\phi(\tau)$ with respect to the distribution (5.24), we get

$$\langle E^*(t+\tau)E(t)\rangle = I_{\mathrm{ss}}\mathrm{e}^{\mathrm{i}\omega_L\tau}\int \mathrm{d}\phi\, P(\phi,\tau|\phi_0)\mathrm{e}^{-\mathrm{i}(\phi-\phi_0)} = I_{\mathrm{ss}}\mathrm{e}^{\mathrm{i}\omega_L\tau}\mathrm{e}^{-D\tau}.$$

The temporal correlation function thus decays exponentially with a coherence time $\tau_c = 1/D$. Taking the Fourier transform — the correct formula is

$$S(\omega) = 2\,\mathrm{Re}\int_0^\infty \mathrm{d}\tau\,\mathrm{e}^{-\mathrm{i}\omega\tau}\langle E^*(t+\tau)E(t)\rangle = \frac{2I_{\mathrm{ss}}D}{(\omega-\omega_L)^2 + D^2}$$

— we find that the laser spectrum is centered at ω_L with a width of the order of D and a Lorentzian lineshape. *The laser linewidth is thus limited by the phase diffusion coefficient.*

Diffusion coefficient

We now have to find an expression for the phase diffusion coefficient. To this end, we shall derive an equation similar to the diffusion equation (5.23). Since this equation deals with a distribution function for the phase, it seems natural to introduce a (quasi-)probability distribution for the laser amplitude. In the last semester, we learned that the P-function does this job: $P(\alpha, \alpha^*)$ gives the (quasi-)probability that a coherent state $|\alpha\rangle$ occurs in an expansion of the density operator in the basis of coherent states:

$$\rho = \int \mathrm{d}^2\alpha \, P(\alpha, \alpha^*)|\alpha\rangle\langle\alpha|. \tag{5.25}$$

What is the equation of motion for this distribution? From the results discussed in subsection 5.3.5 we get the following replacement table for the action of the photon operators on the density operator

$$\begin{aligned} a\rho &\mapsto \alpha P \\ a^\dagger\rho &\mapsto \left(\alpha^* - \frac{\partial}{\partial\alpha}\right) P \\ \rho a &\mapsto \left(\alpha - \frac{\partial}{\partial\alpha^*}\right) P \\ \rho a^\dagger &\mapsto \alpha^* P. \end{aligned} \tag{5.26}$$

The equation resulting from (5.19) is

$$\begin{aligned} \frac{\partial}{\partial t}P(\alpha, \alpha^*) &= \frac{1}{2}\left\{\frac{\partial}{\partial\alpha}(\kappa - G)\alpha + \frac{\partial}{\partial\alpha^*}(\kappa - G)\alpha^*\right\} P \\ &\quad + \frac{G}{4}\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}P, \end{aligned} \tag{5.27}$$

where we have yet neglected gain saturation. (The derivatives act on everything to their right, including P.) An approximate way to take it into account is to replace $G \mapsto G(|\alpha|^2) = G_0/(1 + B|\alpha|^2)$. This is actually an approximation because when transforming to the P-representation, one has to neglect some second order and higher order derivatives.

Let us note that the second-order derivative $\partial_\alpha\partial_{\alpha^*}$ in Eq.(5.27) is directly related to the fact that in the quantum description, the operators a and $a^\dagger$ do not commute. (Their action on a coherent state cannot reduce to

multiplication with the numbers α and α^*, but must involve some derivative, as seen in the replacement rules (5.26)). We already suspect that the second-order derivative may have to do with phase diffusion. We see here that it is connected to the discrete nature of the photons. Sometimes, people develop the picture that each photon that is spontaneously emitted by the gain medium contributes to the cavity field with some "one-photon amplitude" whose phase is arbitrary. The amplitude of the cavity field thus performs a "random walk" in phase space together with a "deterministic" increase related to "stimuled emission" where the additional photons add up "in phase" with the field.

We now have with (5.27) a partial differential equation for a phase space distribution. It features second-order derivatives like the simple diffusion equation (5.23). Since we are interested in phase diffusion, it seems natural to use polar coordinates $\alpha = r\,\mathrm{e}^{\mathrm{i}\phi}$. You are asked to make this transformation in the exercises and to derive the result

$$\frac{\partial}{\partial t}P(r, \phi) = \frac{1}{2r}\frac{\partial}{\partial r}r^2(\kappa - G(r^2))P + \frac{G(r^2)}{4r}\left(\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^2}{\partial\phi^2}\right)P. \tag{5.28}$$

The steady state solution of this equation does not depend on the phase, but only on the modulus r of the laser amplitude. In the weak saturation limit where $G(r^2) \approx G_0(1 - Br^2)$, it is given by

$$P_{\mathrm{ss}}(r) \approx \mathcal{N}\,\exp\left[-\frac{B}{2}\left(r^2 - \frac{G_0 - \kappa}{G_0 B}\right)^2\right].$$

This shows a maximum at $r^2_{\max} = n_{\max} = (G_0 - \kappa)/G_0 B$ where we recover the formula for the semiclassical steady-state intensity (up a conversion factor between intensity and photon number). From this distribution, one can check that the fluctuations of the laser intensity (and the field's modulus) are small in the limit $1/B \gg 1$ (large photon number on average, meaning high above threshold). We can thus confirm that if there are fluctuations, they occur dominantly in the phase of the field.

With this argument, we can go back to the diffusion equation (5.28) and identify the phase diffusion coefficient:

$$D = \frac{G(r^2)}{4r^2} \approx \frac{G(n_{\max})}{4n_{\max}}. \tag{5.29}$$

This formula has been derived first in the 1960/70's by Schawlow and Townes. It shows that phase diffusion (and the laser linewidth), well above threshold, decreases inversely proportional to the laser intensity. When the gain is increased, the emission spectrum thus shows an ever growing peak close to the frequency of the cavity mode, that becomes narrower and narrower. This behaviour is often taken as an experimental proof that a laser is operating.

5.3.5 Operator replacement rules for the P-function

From the master equation for the density operator ρ, we need to find the equation of motion for the P-function. To this end, we use its definition:

$$\rho = \int \mathrm{d}^2\alpha\, |\alpha\rangle\langle\alpha| P(\alpha), \tag{5.30}$$

the density matrix is expanded in projection operators on coherent states. Let us work out the action of the different operators in the master equation on these projectors.

$$\left[a^\dagger a,\, |\alpha\rangle\langle\alpha|\right] = \alpha a^\dagger |\alpha\rangle\langle\alpha| - \alpha^* |\alpha\rangle\langle\alpha| a$$

This was easy. Now we have to get an idea how the creation operator acts on a coherent state. Using its Fock state expansion, we find

$$a^\dagger|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0} \frac{(n+1)\alpha^n}{\sqrt{(n+1)!}} |n+1\rangle = \mathrm{e}^{-|\alpha|^2/2} \frac{\partial}{\partial\alpha} \sum_{n=0} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

We denote this last sum (an un-normalized coherent state) by $\|\alpha\rangle$, following the book by Orszag. Under the α-integral, we get

$$a^\dagger \rho = \int \mathrm{d}^2\alpha\, P(\alpha,\alpha^*)\, \mathrm{e}^{-|\alpha|^2} \frac{\partial}{\partial\alpha} \|\alpha\rangle\langle\alpha\|$$

because $\partial_\alpha \langle\alpha\| = 0$ when α and α^* are regarded as independent variables. We perform one partial integration and discard the boundary terms (they are zero because of the exponential factor and because P should be normalizable):

$$\begin{aligned} a^\dagger \rho &= -\int \mathrm{d}^2\alpha \left(\frac{\partial}{\partial\alpha} \big(P(\alpha,\alpha^*)\, \mathrm{e}^{-|\alpha|^2} \big) \right) \|\alpha\rangle\langle\alpha\| \\ &= \int \mathrm{d}^2\alpha\, |\alpha\rangle\langle\alpha| \left(\alpha^* - \frac{\partial}{\partial\alpha} \right) P(\alpha,\alpha^*). \end{aligned}$$

The factor α^* comes from the differentiation of the gaussian We have switched back to normalized coherent states. Finally, we compare the coefficients in the expansion in coherent states[4] and get the replacement rule

$$a^\dagger \rho \mapsto \left(\alpha^* - \frac{\partial}{\partial\alpha} \right) P(\alpha,\alpha^*). \tag{5.31}$$

In a similar way, one can show that

$$\rho a \mapsto \left(\alpha - \frac{\partial}{\partial\alpha^*} \right) P(\alpha,\alpha^*). \tag{5.32}$$

5.3.6 Coherently pumped and lossy mode: P-function

In the following, we present another example where the P-function can be explicitly constructed. We consider a coherently pumped single-mode field and see how losses relax the mode into a stationary state. We have found before that when the pump continues forever, the mode stays in a coherent state with ever larger amplitude $\alpha = gt$.

The loss of the cavity mode is described by the Liouvillian

$$\dot\rho = \mathcal{L}[\rho] = -\frac{\kappa}{2}\{a^\dagger a,\, \rho\} + \kappa\, a\rho a^\dagger \tag{5.33}$$

where κ is the damping rate. For the coherent driving, we use the Hamiltonian

$$H = -\hbar\Delta a^\dagger a + \mathrm{i}\hbar(g a^\dagger - g^* a)$$

where $\Delta = \omega_p - \omega_m$ is the detuning between the pump and mode frequencies.

From the replacement rules given in Subsection 5.3.5, we find that the commutator with the Hamiltonian maps to

$$\left[a^\dagger a,\, \rho\right] \mapsto -\left(\frac{\partial}{\partial\alpha}\alpha - \frac{\partial}{\partial\alpha^*}\alpha^* \right) P(\alpha,\alpha^*),$$

where the differential operators act on everything to their right, including the P-function.

[4] This uses the fact — not demonstrated her— that the expansion of a state in P-functions is unique.

The other terms in the master equation can be handled in the same way. We find

$$\dot{P}(\alpha) = \frac{\partial}{\partial \alpha}\left[\left(-\mathrm{i}\Delta\alpha - g + \frac{\kappa}{2}\alpha\right) P(\alpha)\right] + \frac{\partial}{\partial \alpha^*}\left[\left(\mathrm{i}\Delta\alpha - g^* + \frac{\kappa}{2}\alpha^*\right) P(\alpha)\right]$$

which is equivalent to the master equation.

Stationary state

Let us look at its solution in steady state. If our intuition can be trusted, we expect the stationary mode amplitude to be finite. The equation can be fulfilled with the following *Ansatz*:

$$\begin{aligned} \left(-\mathrm{i}\Delta\alpha - g + \frac{\kappa}{2}\alpha\right) P(\alpha) &= \text{const.}(\alpha^*) \\ \left(\mathrm{i}\Delta\alpha - g^* + \frac{\kappa}{2}\alpha^*\right) P(\alpha) &= \text{const.}(\alpha) \end{aligned} \tag{5.34}$$

where the constants may depends on α^* or α. Note that the second condition is the complex conjugate of the first. A first guess of solution is given by choosing a nonzero constant, leading to

$$\begin{aligned} P^{(st)}(\alpha) &= \frac{N}{-\mathrm{i}\Delta\alpha - g + \frac{\kappa}{2}\alpha}\,\frac{1}{\mathrm{i}\Delta\alpha^* - g^* + \frac{\kappa}{2}\alpha^*} \\ &= \frac{N}{\Delta^2 + \kappa^2/4}\,\frac{1}{|\alpha - \alpha_s|^2} \qquad \text{(wrong)} \end{aligned}$$

which is peaked at the stationary amplitude given by

$$\alpha_s = \frac{g}{\kappa/2 - \mathrm{i}\Delta} \tag{5.35}$$

But this solution is ill: it is not normalizable because the singularity at $\alpha = \alpha_s$ is too strong.

Our only choice is to choose a constant which is zero in (5.34). This leads us to the condition

$$(\alpha - \alpha_s) P(\alpha) = 0$$

that also fulfills the complex conjugate condition. The solution of this equation is . . . a $\delta^{(2)}$-function. With the correct normalization, we have

$$P^{(st)}(\alpha) = \delta^{(2)}(\alpha - \alpha_s). \tag{5.36}$$

The field mode thus ends up in a coherent state with an amplitude given by the ratio between the pump rate g and the loss rate κ. This is typical for a steady state solution where “gain” is compensated by “loss”, and we shall encounter other examples in laser theory.

We also note that the “zero constant” option we were forced to take also occurs generically. One can show that this is the only possibility to obtain a normalizable solution.