Einführung in die Quantenoptik Wintersemester 2007/08

Carsten Henkel

Problem set 3

Hand out: 09 Nov 2007
Hand in: 23 Nov 2007

Problem 3.1 – Commutators of free fields (6 points)

In the lecture, we have found that the commutator between the quantized vector potential $\mathbf{A}$ and its conjugate momentum field $\mathbf{\Pi}$ should be

$$[A_i(\mathbf{x}), \Pi_j(\mathbf{x}')] = \mathrm{i}\hbar\delta^{\perp}_{ij}(\mathbf{x} - \mathbf{x}') \tag{3.1}$$

(i) From the classical theory, you know that the momentum $\mathbf{\Pi}$ is related to the 'velocity' $\dot{\mathbf{A}}$. This remains true in the quantum theory. Use the relation to write a mode expansion for $\mathbf{\Pi}$ and for the electric field $\mathbf{E}$.

(ii) Take the divergence of Eq.(3.1) with respect to $\mathbf{x}'$ and the index j and comment on why you get zero.

(iii) Take the rotation with respect to $\mathbf{x}$ and observe that this is nonzero. Show that one gets the so-called Pauli commutator between the (transverse) fields $\mathbf{E}$ and $\mathbf{B}$:

$$[B_i(\mathbf{x}), E_j(\mathbf{x}')] = \mathrm{i}\hbar\varepsilon_0\epsilon_{ijk}\frac{\partial}{\partial x_k}\delta(\mathbf{x} - \mathbf{x}') \tag{3.2}$$

(iv) (3 bonus points) In a mathematically careful (quantum) field theory, the fields $\mathbf{A}$, $\mathbf{E}$, $\mathbf{B}$ etc are actually (operator-valued) distributions that have to be 'smeared out' with suitable smooth test functions $\mathbf{f}(\mathbf{x})$, $\mathbf{g}(\mathbf{x})$. This gives operators that behave in a less singular way. Consider thus the observable

$$\mathcal{E}[\mathbf{f}] = \int \mathrm{d}^3x\, \mathbf{f}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}) \tag{3.3}$$

and similarly for $\mathcal{B}[\mathbf{g}]$. Work out the commutator $[\mathcal{E}[\mathbf{f}], \mathcal{B}[\mathbf{g}]]$ and conclude that orthogonal components of $\mathbf{E}$ and $\mathbf{B}$ at neighboring points cannot be measured simultaneously. Derive, as in the quantum mechanics I course, the uncertainty relation between the variances $(\Delta\mathcal{E}[\mathbf{f}])^2$, $(\Delta\mathcal{B}[\mathbf{g}])^2$.

(v) (2 bonus points) In the construction of (iv), focus on mode functions of the form $\nabla \times \mathbf{g} = (\omega/c)\mathbf{f}$ with $\mathbf{f}$ being normalized as $\int \mathrm{d}^3x\, \mathbf{f}^2(\mathbf{x}) = 1$ and $\omega > 0$ a constant. Consider a quantum state for the field where the average values of $\mathcal{E}[\mathbf{f}]$, $\mathcal{B}[\mathbf{g}]$ are zero and their variances are identical. What do you get then for the 'energy density' $\varepsilon_0\langle\mathcal{E}[\mathbf{f}]^2\rangle + (1/\mu_0)\langle\mathcal{B}[\mathbf{g}]^2\rangle$?

Answer: One 'photon energy' $\hbar\omega$ spread over the spatial size of the test function $\mathbf{f}(\mathbf{x})$.

Problem 3.2 – Finite size corrections to the vacuum energy (Casimir force) (8 points)

In a one-dimensional cavity where the electromagnetic fields satisfy Dirichlet boundary conditions at $x = 0$ and $x = L$, the mode frequencies are given by (you have found this in Problem 2.2):

$$\omega_k = ck, \qquad k = \frac{\pi}{L}, \frac{2\pi}{L}, \frac{3\pi}{L}, \ldots \tag{3.4}$$

In this exercise you compare the corresponding vacuum energy

$$E(L) = \sum_k \frac{\hbar\omega_k}{2} \tag{3.5}$$

to its value in an infinitely extended cavity. For simplicity, we ignore here the polarization (that would give an extra factor of two). You know from the lecture that the vacuum energy density u in free space tends to a (infinite) constant if the 'quantization box' becomes large. In 1D, the (divergent) expression for this energy density is

$$u = \int \frac{\mathrm{d}k}{2\pi} \frac{\hbar c|k|}{2} = \int_0^\infty \frac{\mathrm{d}k}{\pi} \frac{\hbar ck}{2} \tag{3.6}$$

We now evaluate the energy difference between the actual vacuum energy in the cavity and the vacuum energy in the same region of space (length L) when the cavity walls are removed,

$$E_{\mathrm{Cas}}(L) = E(L) - Lu. \tag{3.7}$$

This is called the Casimir energy and turns out to be a finite quantity.

(i) Show that $E_{\mathrm{Cas}}(L)$ can be reduced to

$$E_{\mathrm{Cas}}(L) = \alpha \frac{\hbar c}{L} \Big(\sum_{x=0}^{\infty} f(x) - \int_0^\infty \mathrm{d}x\, f(x) \Big) \tag{3.8}$$

where the dimensionless constant α and the function $f(x)$ remains to be determined.

(ii) Both the sum and the integral in Eq.(3.8) diverge at the upper limit. To get finite quantities, we replace $f(x)$ by $f(x)\Lambda(x)$ where the 'cutoff function' is flat up to some cutoff $x_c \gg 1$ and decays to zero sufficiently fast to ensure convergence:

$$\Lambda(0) = 1, \qquad \Lambda'(0) = 0, \qquad \lim_{x \gg x_c} \Lambda(x) = 0 \tag{3.9}$$

Use the Euler-MacLaurin formula (see mathematics textbooks) to find that in the limit $x_c \to \infty$, one gets

$$E_{\mathrm{Cas}}(L) = \alpha' \frac{\hbar c}{L} \tag{3.10}$$

with another dimensionless constant α' that you are invited to calculate.

Problem 3.3 – Blackbody and vacuum energy (spectrum) (6 points)
The electromagnetic energy density in vacuum at temperature T is known as the Planck formula. It can be written in the form

$$u(T) = \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \frac{2\hbar\omega^3}{c^3} \Big(\frac{1}{\mathrm{e}^{\beta\omega} - 1} + \frac{1}{2} \Big) \tag{3.11}$$

where $\beta\omega = \hbar\omega/k_\mathrm{B}T$ and the term $\frac{1}{2}$ accounts for the vacuum energy. Eq.(3.11) can be derived from the quantized field by considering a so-called 'thermal state'; this is what you do in thie exercise.

(i) The concept of a thermal state combines stationary states with classical statistics in the canonical ensemble. Each stationary state $|\{n\}\rangle$ with energy $E_{\{n\}}$ appears with a weight given by the Boltzmann factor $\exp(-E_{\{n\}}/k_\mathrm{B}T)$. The notation $\{n\}$ stands for the set of occupation (photon) numbers in all field modes. Show first that for a stationary state, one has

$$\langle\{n\}|a_k^\dagger a_l|\{n\}\rangle = \delta_{kl} n_k, \qquad \langle\{n\}|a_k a_l^\dagger|\{n\}\rangle = \delta_{kl}(n_k + 1) \tag{3.12}$$

where n_k is the photon number in mode k. Show also that all other products of one or two mode operators have zero expectation value. We use in the following that in a thermal state, these expectation values are changed into

$$\langle a_k^\dagger a_l\rangle_T = \delta_{kl}\bar{n}(\omega_k), \qquad \langle a_k a_l^\dagger\rangle = \delta_{kl}(\bar{n}(\omega_k) + 1) \tag{3.13}$$

where $\bar{n}(\omega_k) = (\mathrm{e}^{\beta\omega_k} - 1)^{-1}$ is mean photon number for a mode at frequency ω_k.

(ii) Start from the mode expansion of the electric and magnetic field operators in a plane wave basis and work out the operator for the electromagnetic energy density u in terms of the a_k and $a_k^\dagger$. Take the average of u in a thermal state, using the relations (3.13) and get

$$u(T) := \langle u\rangle_T = \sum_k \frac{2\hbar\omega_k}{V}(\bar{n}(\omega_k) + \tfrac{1}{2}) \tag{3.14}$$

Take the 'continuum limit' $V \to \infty$ where the sum over $\mathbf{k}$ is replaced by an integral to get the Planck-formula (3.11).

(iii) Focus on $T = 0$, cut off the frequency integral by some reasonably chosen value (the largest mass of a known elementary particle, the Planck energy) and find a numerical estimate for the 'vacuum energy density'. What cutoff frequency would give an energy density equal to the density of dark energy (about the critical mass density $\sim 10^{-27}\mathrm{kg/m^3}$ times c^2)?