Einführung in die Quantenoptik Wintersemester 2007/08

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Problem set 4 Hand out: 23 Nov 2007

Hand in: 07 Dec 2007

Problem 4.1 – Unitary operators (4 points)

We have seen three examples of unitary operators U on the Hilbert space for a single field mode: the free time evolution operator $\exp(-\mathrm{i}a^\dagger a\omega t)$, the displacement operator $\exp(\alpha a^\dagger - \alpha^* a)$ and the squeezing operator $\exp(\xi a^{\dagger 2} - \xi^* a^2)$. The action of U on any photon mode operator A (a function of a and $a^\dagger$) is given by $A \mapsto U^\dagger A U$.

(a) Show that all these operators preserve the commutation relation between a and $a^\dagger$, i.e., $[a, a^\dagger] \mapsto \mathbb{1}$.

(b) An infinitesimal unitary operation is 'generated' by a hermitean operator J, $U \approx \mathbb{1} - \mathrm{i}\epsilon J$ with ϵ an infinitesimal real parameter (for example: $\omega \mathrm{d}t$). Show that the infinitesimal action on operators involves the commutator $[A, J]$:

$$A \mapsto A - \mathrm{i}\epsilon[A, J] \tag{4.1}$$

and prove that this action leaves commutators unchanged to first order in ϵ: $[A, B] \mapsto [A, B]$.

Problem 4.2 – Coherent states (8 points)

(a) Compute the mean and the variance of the quadrature X_θ (as defined in the lecture) in a coherent state α:

$$\langle X_\theta \rangle_\alpha = \frac{\mathrm{e}^{-\mathrm{i}\theta}\alpha + \mathrm{e}^{\mathrm{i}\theta}\alpha^*}{\sqrt{2}}, \qquad (\Delta X_\theta^2)_\alpha = \frac{1}{2} \tag{4.2}$$

Work out the Heisenberg uncertainty relation between X_θ and $X_{\theta+\pi/2}$ and conclude that a coherent state is a minimum uncertainty state with respect to the quadrature operators.

(b) Compute the commutator between the photon number operator n and the quadrature X_θ. What does this result imply for the variances of photon number and quadrature? Discuss the cases of a Fock (number) state and a Glauber (coherent) state.

(c) In the lecture, we have seen that the probability distribution for the photon number (the 'photon statistics') in a coherent state is given by the Poisson law. As an alternative approach, compute the generating function for the moments of the photon statistics, i.e.,

$$f_\alpha(\xi) = \langle \mathrm{e}^{\mathrm{i}\xi\hat{n}} \rangle_\alpha \tag{4.3}$$

This function also contains the entire information on the probability distribution. Compute the expansion of $\log f_\alpha(\xi)$ up to the third order in ξ. The coefficients give the "cumulants" of the photon statistics (look up `http://de.wikipedia.org/wiki/Kumulante` for the definition of cumulants).

(d) The displacement operators $D(\alpha)$ implement a unitary, projective representation of the additive group in the complex numbers on the Hilbert space of a harmonic oscillator. To prove this statement, you have to show that

$$D(\alpha)D(\beta) = \mathrm{e}^{\mathrm{i}\varphi(\alpha,\beta)} D(\alpha + \beta) \tag{4.4}$$

where the phase $\varphi(\alpha, \beta)$ remains to be computed. Do displacement operators for $\alpha \neq \beta$ commute? If not, does it really matter?

Problem 4.3 – Squeezed states (8 points)

(a) In the lecture, we have seen that the squeezing operator $S(\xi) = \exp(\xi a^{\dagger 2} - \xi^* a^2)$ has the property

$$S^\dagger(\xi) X S(\xi) = X\,\mathrm{e}^{2|\xi|} \tag{4.5}$$

if $X = X_0$ is the 'position quadrature' and ξ is real. Generalize the calculation to complex-valued ξ and a suitable quadrature X_θ.

(b) Use this property to show that the squeezed state $|\xi\rangle = S(\xi)|0\rangle$ is annihilated by an operator of the form

$$\left(\eta a + \mu a^\dagger\right)|\xi\rangle = 0 \tag{4.6}$$

where η and μ have to be computed. Show that up to a normalization factor, the ratio $|\eta/\mu|$ is proportional to a hyperbolic tangent, $\tanh(2|\xi|)$.

(c) Conclude that the expansion of the squeezed state in the number state basis is of the form mentioned in the lecture,

$$|\xi\rangle = N \sum_m c_m \tanh^m(2|\xi|)|2m\rangle \tag{4.7}$$

and compute the coefficients c_m.

(d) The 'normal-ordered form' of the squeezing operator is given by (5 bonus points for a proof; maybe error in the phase)

$$\begin{aligned} S(\xi) &= \exp\left(\frac{\nu}{2\mu} a^{\dagger 2}\right) \exp\left(-(a^\dagger a + \tfrac{1}{2}) \log \mu\right) \exp\left(-\frac{\nu^*}{2\mu} a^2\right), && (4.8)\\ \nu &= \mathrm{e}^{\mathrm{i} \arg \xi} \sinh(2|\xi|) && (4.9)\\ \mu &= \cosh(2|\xi|) && (4.10) \end{aligned}$$

Starting from this identity, compute the Q-function for the squeezed state $|\xi\rangle$, defined by

$$Q_\xi(\alpha) = \pi^{-1} \left|\langle\alpha|\xi\rangle\right|^2 \tag{4.11}$$

where $|\alpha\rangle$ is the usual coherent state. Your result gives a geometric interpretation of the "squeezed vacuum" described by the state $|\xi\rangle$.