Einführung in die Quantenoptik Wintersemester 2007/08

Carsten Henkel

Problem set 1

Hand out: 19 Oct 2007
Hand in: 26 Oct 2007

Problem 1.1 – Electromagnetic momentum conservation (8 points)

Prove from the 'macroscopic Maxwell equations' (i.e., the equations for the fields $\mathbf{E}, \mathbf{B}, \mathbf{D}, \mathbf{H}$ with given source terms ρ, $\mathbf{j}$) that the following conservation law holds

$$\partial_t g_i + \partial_j T_{ji} = f_i \tag{1.1}$$

with the momentum density and momentum current (or 'Minkowski stress tensor')

$$\mathbf{g} = \mathbf{D} \times \mathbf{B} \tag{1.2}$$

$$T_{ji} = D_j E_i - \delta_{ji} \frac{\mathbf{D} \cdot \mathbf{E}}{2} + H_j B_i - \delta_{ji} \frac{\mathbf{H} \cdot \mathbf{B}}{2} \tag{1.3}$$

Feel free to make special assumptions on the material equations (nondispersive, linear, isotropic etc.) to arrive at this result. Determine the force density $\mathbf{f}$ and comment on the relation to the Coulomb-Lorentz force.

Problem 1.2 – The Hamiltonian of relativistic electrodynamics (7 points)

Show that the following Hamiltonian for a relativistic point charge in a given electromagnetic field $\mathbf{A}$, ϕ leads to the correct equations of motion

$$H = \sqrt{[\mathbf{p} - e\mathbf{A}(\mathbf{r})]^2 c^2 + m^2 c^4} + e\phi(\mathbf{r}) \tag{1.4}$$

where $\mathbf{r}$ is the coordinate and $\mathbf{p}$ the canonically conjugate momentum.

Hint. Use the canonical equations of motion: $\dot{\mathbf{p}} = -\partial H / \partial \mathbf{r}$ etc.

Problem 1.3 – Conventional laser fields on the atomic scale (5 points)

Take the most powerful laser you (or your experimentalist colleagues) have in the lab and make an estimate of the electric field in the best focus you can achieve. Compare this field to the electric field in the Hydrogen atom (take a distance of one Bohr radius from the nucleus).

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Problem set 2 Hand out: 26 Oct 2007
Hand in: 09 Nov 2007

Problem 2.1 – Transverse δ-function (7 points)

In the lecture, we have encountered the so-called 'transverse δ-function $\delta^{\perp}_{ij}(\mathbf{x})$. It has the following properties:

$$F^{\perp}_i(\mathbf{x}) := \int \mathrm{d}^3x' \delta^{\perp}_{ij}(\mathbf{x}-\mathbf{x}') F_j(\mathbf{x}') \quad \text{is a transverse vector field} \tag{2.1}$$

$$\int \mathrm{d}^3x' \delta^{\perp}_{ij}(\mathbf{x}-\mathbf{x}') F_j(\mathbf{x}') = \begin{cases} F_i(\mathbf{x}) & \text{if } \mathbf{F}(\mathbf{x}) \text{ is transverse} \\ 0 & \text{if } \mathbf{F}(\mathbf{x}) = \nabla\phi(\mathbf{x}) \end{cases} \tag{2.2}$$

$$\delta^{\perp}_{ij}(\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} (\delta_{ij} - \frac{k_i k_j}{k^2}) \tag{2.3}$$

$$\delta^{\perp}_{ij}(\mathbf{x}) = \delta_{ij}\delta(\mathbf{x}) + \frac{1}{4\pi} \frac{\partial^2}{\partial x_i \partial x_j} \frac{1}{|\mathbf{x}|} \tag{2.4}$$

(1) Describe in words the meaning of (2.3).

(2) Show that Eq.(2.3) correctly implements property (2.2) by working with the spatial Fourier transform of the vector function $\mathbf{F}(\mathbf{x})$.

(3) Prove Eq.(2.4) by starting from Eq.(2.3).

(4) Find a second proof of Eq.(2.4) by analogy to a gauge transformation: Be $\mathbf{F}(\mathbf{x})$ any (sufficiently smooth) vector field. Consider it as a vector potential and find a gauge transformation such that the transformed $\mathbf{F}'(\mathbf{x})$ is transverse.

Hint. Observe that the 'gauge function' solves an inhomogeneous Laplace equation.

Problem 2.2 – Field quantization in a simple geometry (8 points)

Consider a cavity bounded by two parallel mirrors where the mode functions for the electromagnetic field take the form

$$\mathbf{f}_{\kappa}(\mathbf{x}) = N\mathbf{e} f(x) \tag{2.5}$$

where N is a normalization factor, x the coordinate perpendicular to the mirrors and $\mathbf{e}$ a spatially constant polarization vector.

(1) Find $\mathbf{e}$ such the mode function is transverse.

(2) Assume periodic boundary conditions on the interval $0 \le x \le L$ and construct a real-valued $f(x)$ that solves the Helmholtz equation.* Find N and an integration domain for the y, z coordinates such that the mode function is normalized as in the lecture.

* The Helmholtz equation is

$$\nabla^2 f + \frac{\omega^2}{c^2} f = 0 \tag{2.6}$$

(3) What is the eigenfrequency ω_κ of this mode? What changes if you adopt boundary conditions for a cavity with perfectly reflecting walls at $x = 0, L$?

(4) Continue with periodic boundary conditions and real-valued modes and bring the total field momentum

$$\mathbf{P} = \varepsilon_0 \int \mathrm{d}^3x \, \mathbf{E} \times \mathbf{B} \qquad \text{(wrong; please symmetrize)} \tag{2.7}$$

in the form of a sum over mode indices κ and annihilation and creation operators a_κ, $a_\kappa^\dagger$. You can use the mode expansion from the lecture for the vector potential to compute the fields. Try to give an interpretation of the result that the momentum density is not a simple sum over modes.

Solution. We discussed in the exercise session the special case of the field momentum for cavity with perfectly reflecting walls [part (3)]. The expansion for the electric field can be obtained taking the time derivative of the vector potential, and reads

$$\mathbf{E}(x) = \mathrm{i}\mathbf{e} \sum_k \sqrt{\frac{\hbar\omega_k}{\varepsilon_0 AL}} (a_k - a_k^\dagger) \sin(kx) \tag{2.8}$$

where the discrete k-vectors take the values $k = \pi/L, 2\pi/L, \ldots$. The magnetic field is given by

$$\mathbf{B}(x) = \mathbf{e}_x \times \mathbf{e} \sum_{k'} \sqrt{\frac{\hbar}{\varepsilon_0 \omega_{k'} AL}} (a_{k'} + a_{k'}^\dagger) k' \cos(k'x) \tag{2.9}$$

The field momentum is actually not correctly defined in Eq.(2.7): the operators $\mathbf{E}$ and $\mathbf{B}$ do not commute, and $\mathbf{P}$ would not be a hermitean operator. We symmetrize the product to get a hermitean momentum. To proceed, we need the following results

$$\begin{aligned}
&\mathbf{e} \times (\mathbf{e}_x \times \mathbf{e}) = \mathbf{e}_x \\
&(a_k - a_k^\dagger)(a_{k'} + a_{k'}^\dagger) + (a_{k'} + a_{k'}^\dagger)(a_k - a_k^\dagger) = \text{symmetric under } k \leftrightarrow k' \qquad (2.10) \\
&\int_0^L \mathrm{d}x \, \sin(kx) k' \cos(k'x) = \int_0^L \mathrm{d}x \, \sin(kx) \frac{\partial}{\partial x} \sin(k'x) \\
&= -\int_0^L \mathrm{d}x \, k \cos(kx) \sin(k'x) \qquad (2.11)
\end{aligned}$$

Eq.(2.11) is found by partial integration. The integrated terms involve $\sin(kx)\sin(k'x)$ on the mirrors $x = 0, L$ and vanish there. Hence, this part is anti-symmetric under the $k \leftrightarrow k'$. (The special case $k = k'$ gives zero.)

Finally, we observe that the sum over k and k' involves a product of symmetric and anti-symmetric functions of k and k'. Hence, we get zero. The same result is obtained with periodic boundary conditions ($k = 0, 2\pi/L, 4\pi/L, \ldots$) and real mode functions $\sin(kx)$ and $\cos(kx)$.

It may still be asked how one can ever get a nonzero field momentum in a perfectly reflecting cavity. A physical situation where one would expect a nonzero momentum, at least temporarily, is a 'Q-switched laser' where a wavepacket (created by a superposition of modes) circulates in the cavity. This wavepacket may well be 'on its way to the right mirror' in the middle of the cavity and carry momentum like a laser beam or any plane plane wave does. For the moment, it is not clear what quantum state for the mode operators a_k has to be chosen to get a nonzero momentum in this situation.

Problem 2.3 – The electric field per photon (5 points)

In the lecture, we have found the following mode expansion for the (transverse) vector potential (operator) in a 'quantization box' of volume V

$$\mathbf{A}(\mathbf{x}, t) = \sum_{\kappa} \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_\kappa V}} \left(\mathbf{e}_\kappa \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega_\kappa t)} a_\kappa + \text{h.c.} \right). \tag{2.12}$$

where the label κ combines the wavevector and polarization of a given mode, $\mathbf{e}_\kappa$ is the polarization vector, and ω_κ is the mode frequency.

(1) Derive the electric field operator.

(2) In the following, we use this expansion as a rough estimate for the quantized field in a cavity. Take a quantization box with length L (distance between mirrors) and transverse area A (size of a focused beam). Focus on one mode and give an estimate for the 'electric field per photon' at the cavity centre. (I.e.: the amplitude of the mode function that multiplies the operator a_k or $a_k^\dagger$.)

(3) Take an atom of the size of the Hydrogen atom and estimate the electric dipole interaction energy V_{dip} for a mode nearly resonant with a typical transition between Hydrogen bound states. You can use that the electric dipole moment d is of the order of atomic size times electron charge. Compare this interaction energy to (i) the binding energy of Hydrogen and to (ii) the natural linewidth $\hbar\gamma = d^2 k^3 / 3\pi\varepsilon_0$ of a transition with wavenumber k. In the case that $V_{\text{dip}} > \hbar\gamma$, one says that a single photon can 'saturate' this transition. What cavity or atomic parameters are needed to enter this regime?

Einführung in die Quantenoptik Wintersemester 2007/08

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Problem set 3 Hand out: 09 Nov 2007

Hand in: 23 Nov 2007

Problem 3.1 – Commutators of free fields (6 points)

In the lecture, we have found that the commutator between the quantized vector potential $\mathbf{A}$ and its conjugate momentum field $\mathbf{\Pi}$ should be

$$[A_i(\mathbf{x}), \Pi_j(\mathbf{x}')] = \mathrm{i}\hbar\delta^{\perp}_{ij}(\mathbf{x} - \mathbf{x}') \tag{3.1}$$

(i) From the classical theory, you know that the momentum $\mathbf{\Pi}$ is related to the 'velocity' $\dot{\mathbf{A}}$. This remains true in the quantum theory. Use the relation to write a mode expansion for $\mathbf{\Pi}$ and for the electric field $\mathbf{E}$.

(ii) Take the divergence of Eq.(3.1) with respect to $\mathbf{x}'$ and the index j and comment on why you get zero.

(iii) Take the rotation with respect to $\mathbf{x}$ and observe that this is nonzero. Show that one gets the so-called Pauli commutator between the (transverse) fields $\mathbf{E}$ and $\mathbf{B}$:

$$[B_i(\mathbf{x}), E_j(\mathbf{x}')] = \mathrm{i}\hbar\varepsilon_0\epsilon_{ijk}\frac{\partial}{\partial x_k}\delta(\mathbf{x} - \mathbf{x}') \tag{3.2}$$

(iv) (3 bonus points) In a mathematically careful (quantum) field theory, the fields $\mathbf{A}$, $\mathbf{E}$, $\mathbf{B}$ etc are actually (operator-valued) distributions that have to be 'smeared out' with suitable smooth test functions $\mathbf{f}(\mathbf{x})$, $\mathbf{g}(\mathbf{x})$. This gives operators that behave in a less singular way. Consider thus the observable

$$\mathcal{E}[\mathbf{f}] = \int \mathrm{d}^3x\, \mathbf{f}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}) \tag{3.3}$$

and similarly for $\mathcal{B}[\mathbf{g}]$. Work out the commutator $[\mathcal{E}[\mathbf{f}], \mathcal{B}[\mathbf{g}]]$ and conclude that orthogonal components of $\mathbf{E}$ and $\mathbf{B}$ at neighboring points cannot be measured simultaneously. Derive, as in the quantum mechanics I course, the uncertainty relation between the variances $(\Delta\mathcal{E}[\mathbf{f}])^2$, $(\Delta\mathcal{B}[\mathbf{g}])^2$.

(v) (2 bonus points) In the construction of (iv), focus on mode functions of the form $\nabla \times \mathbf{g} = (\omega/c)\mathbf{f}$ with $\mathbf{f}$ being normalized as $\int \mathrm{d}^3x\, \mathbf{f}^2(\mathbf{x}) = 1$ and $\omega > 0$ a constant. Consider a quantum state for the field where the average values of $\mathcal{E}[\mathbf{f}]$, $\mathcal{B}[\mathbf{g}]$ are zero and their variances are identical. What do you get then for the 'energy density' $\varepsilon_0\langle\mathcal{E}[\mathbf{f}]^2\rangle + (1/\mu_0)\langle\mathcal{B}[\mathbf{g}]^2\rangle$?

Answer: One 'photon energy' $\hbar\omega$ spread over the spatial size of the test function $\mathbf{f}(\mathbf{x})$.

Problem 3.2 – Finite size corrections to the vacuum energy (Casimir force) (8 points)

In a one-dimensional cavity where the electromagnetic fields satisfy Dirichlet boundary conditions at $x = 0$ and $x = L$, the mode frequencies are given by (you have found this in Problem 2.2):

$$\omega_k = ck, \qquad k = \frac{\pi}{L}, \frac{2\pi}{L}, \frac{3\pi}{L}, \ldots \tag{3.4}$$

In this exercise you compare the corresponding vacuum energy

$$E(L) = \sum_k \frac{\hbar\omega_k}{2} \tag{3.5}$$

to its value in an infinitely extended cavity. For simplicity, we ignore here the polarization (that would give an extra factor of two). You know from the lecture that the vacuum energy density u in free space tends to a (infinite) constant if the 'quantization box' becomes large. In 1D, the (divergent) expression for this energy density is

$$u = \int \frac{\mathrm{d}k}{2\pi} \frac{\hbar c|k|}{2} = \int\limits_0^\infty \frac{\mathrm{d}k}{\pi} \frac{\hbar ck}{2} \tag{3.6}$$

We now evaluate the energy difference between the actual vacuum energy in the cavity and the vacuum energy in the same region of space (length L) when the cavity walls are removed,

$$E_{\mathrm{Cas}}(L) = E(L) - Lu. \tag{3.7}$$

This is called the Casimir energy and turns out to be a finite quantity.

(i) Show that $E_{\mathrm{Cas}}(L)$ can be reduced to

$$E_{\mathrm{Cas}}(L) = \alpha\frac{\hbar c}{L}\Big(\sum_{x=0}^{\infty} f(x) - \int\limits_0^\infty \mathrm{d}x\, f(x)\Big) \tag{3.8}$$

where the dimensionless constant α and the function $f(x)$ remains to be determined.

(ii) Both the sum and the integral in Eq.(3.8) diverge at the upper limit. To get finite quantities, we replace $f(x)$ by $f(x)\Lambda(x)$ where the 'cutoff function' is flat up to some cutoff $x_c \gg 1$ and decays to zero sufficiently fast to ensure convergence:

$$\Lambda(0) = 1, \qquad \Lambda'(0) = 0, \qquad \lim_{x \gg x_c} \Lambda(x) = 0 \tag{3.9}$$

Use the Euler-MacLaurin formula (see mathematics textbooks) to find that in the limit $x_c \to \infty$, one gets

$$E_{\mathrm{Cas}}(L) = \alpha'\frac{\hbar c}{L} \tag{3.10}$$

with another dimensionless constant α' that you are invited to calculate.

Problem 3.3 – Blackbody and vacuum energy (spectrum) (6 points)
The electromagnetic energy density in vacuum at temperature T is known as the Planck formula. It can be written in the form

$$u(T) = \int_0^\infty \frac{\mathrm{d}\omega}{2\pi} \frac{2\hbar\omega^3}{c^3} \Big(\frac{1}{\mathrm{e}^{\beta\omega} - 1} + \frac{1}{2} \Big) \tag{3.11}$$

where $\beta\omega = \hbar\omega / k_\mathrm{B} T$ and the term $\frac{1}{2}$ accounts for the vacuum energy. Eq.(3.11) can be derived from the quantized field by considering a so-called 'thermal state'; this is what you do in thie exercise.

(i) The concept of a thermal state combines stationary states with classical statistics in the canonical ensemble. Each stationary state $|\{n\}\rangle$ with energy $E_{\{n\}}$ appears with a weight given by the Boltzmann factor $\exp(-E_{\{n\}}/k_\mathrm{B}T)$. The notation $\{n\}$ stands for the set of occupation (photon) numbers in all field modes. Show first that for a stationary state, one has

$$\langle\{n\}| a_k^\dagger a_l |\{n\}\rangle = \delta_{kl} n_k, \qquad \langle\{n\}| a_k a_l^\dagger |\{n\}\rangle = \delta_{kl}(n_k + 1) \tag{3.12}$$

where n_k is the photon number in mode k. Show also that all other products of one or two mode operators have zero expectation value. We use in the following that in a thermal state, these expectation values are changed into

$$\langle a_k^\dagger a_l \rangle_T = \delta_{kl} \bar{n}(\omega_k), \qquad \langle a_k a_l^\dagger \rangle = \delta_{kl}(\bar{n}(\omega_k) + 1) \tag{3.13}$$

where $\bar{n}(\omega_k) = (\mathrm{e}^{\beta\omega_k} - 1)^{-1}$ is mean photon number for a mode at frequency ω_k.

(ii) Start from the mode expansion of the electric and magnetic field operators in a plane wave basis and work out the operator for the electromagnetic energy density u in terms of the a_k and $a_k^\dagger$. Take the average of u in a thermal state, using the relations (3.13) and get

$$u(T) := \langle u \rangle_T = \sum_k \frac{2\hbar\omega_k}{V} (\bar{n}(\omega_k) + \tfrac{1}{2}) \tag{3.14}$$

Take the 'continuum limit' $V \to \infty$ where the sum over $\mathbf{k}$ is replaced by an integral to get the Planck-formula (3.11).

(iii) Focus on $T = 0$, cut off the frequency integral by some reasonably chosen value (the largest mass of a known elementary particle, the Planck energy) and find a numerical estimate for the 'vacuum energy density'. What cutoff frequency would give an energy density equal to the density of dark energy (about the critical mass density $\sim 10^{-27}\mathrm{kg/m}^3$ times c^2)?

Einführung in die Quantenoptik Wintersemester 2007/08

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Problem set 4 Hand out: 23 Nov 2007

Hand in: 07 Dec 2007

Problem 4.1 – Unitary operators (4 points)

We have seen three examples of unitary operators U on the Hilbert space for a single field mode: the free time evolution operator $\exp(-\mathrm{i}a^\dagger a\omega t)$, the displacement operator $\exp(\alpha a^\dagger - \alpha^* a)$ and the squeezing operator $\exp(\xi a^{\dagger 2} - \xi^* a^2)$. The action of U on any photon mode operator A (a function of a and $a^\dagger$) is given by $A \mapsto U^\dagger A U$.

(a) Show that all these operators preserve the commutation relation between a and $a^\dagger$, i.e., $[a, a^\dagger] \mapsto \mathbb{1}$.

(b) An infinitesimal unitary operation is 'generated' by a hermitean operator J, $U \approx \mathbb{1} - \mathrm{i}\epsilon J$ with ϵ an infinitesimal real parameter (for example: $\omega \mathrm{d}t$). Show that the infinitesimal action on operators involves the commutator $[A, J]$:

$$A \mapsto A - \mathrm{i}\epsilon[A, J] \tag{4.1}$$

and prove that this action leaves commutators unchanged to first order in ϵ: $[A, B] \mapsto [A, B]$.

Problem 4.2 – Coherent states (8 points)

(a) Compute the mean and the variance of the quadrature X_θ (as defined in the lecture) in a coherent state α:

$$\langle X_\theta \rangle_\alpha = \frac{\mathrm{e}^{-\mathrm{i}\theta}\alpha + \mathrm{e}^{\mathrm{i}\theta}\alpha^*}{\sqrt{2}}, \qquad (\Delta X_\theta^2)_\alpha = \frac{1}{2} \tag{4.2}$$

Work out the Heisenberg uncertainty relation between X_θ and $X_{\theta+\pi/2}$ and conclude that a coherent state is a minimum uncertainty state with respect to the quadrature operators.

(b) Compute the commutator between the photon number operator n and the quadrature X_θ. What does this result imply for the variances of photon number and quadrature? Discuss the cases of a Fock (number) state and a Glauber (coherent) state.

(c) In the lecture, we have seen that the probability distribution for the photon number (the 'photon statistics') in a coherent state is given by the Poisson law. As an alternative approach, compute the generating function for the moments of the photon statistics, i.e.,

$$f_\alpha(\xi) = \langle \mathrm{e}^{\mathrm{i}\xi\hat{n}} \rangle_\alpha \tag{4.3}$$

This function also contains the entire information on the probability distribution. Compute the expansion of $\log f_\alpha(\xi)$ up to the third order in ξ. The coefficients give the "cumulants" of the photon statistics (look up `http://de.wikipedia.org/wiki/Kumulante` for the definition of cumulants).

(d) The displacement operators $D(\alpha)$ implement a unitary, projective representation of the additive group in the complex numbers on the Hilbert space of a harmonic oscillator. To prove this statement, you have to show that

$$D(\alpha)D(\beta) = \mathrm{e}^{\mathrm{i}\varphi(\alpha,\beta)} D(\alpha + \beta) \tag{4.4}$$

where the phase $\varphi(\alpha, \beta)$ remains to be computed. Do displacement operators for $\alpha \neq \beta$ commute? If not, does it really matter?

Problem 4.3 – Squeezed states (8 points)

(a) In the lecture, we have seen that the squeezing operator $S(\xi) = \exp(\xi a^{\dagger 2} - \xi^* a^2)$ has the property

$$S^\dagger(\xi) X S(\xi) = X\,\mathrm{e}^{2|\xi|} \tag{4.5}$$

if $X = X_0$ is the 'position quadrature' and ξ is real. Generalize the calculation to complex-valued ξ and a suitable quadrature X_θ.

(b) Use this property to show that the squeezed state $|\xi\rangle = S(\xi)|0\rangle$ is annihilated by an operator of the form

$$\left(\eta a + \mu a^\dagger\right)|\xi\rangle = 0 \tag{4.6}$$

where η and μ have to be computed. Show that up to a normalization factor, the ratio $|\eta/\mu|$ is proportional to a hyperbolic tangent, $\tanh(2|\xi|)$.

(c) Conclude that the expansion of the squeezed state in the number state basis is of the form mentioned in the lecture,

$$|\xi\rangle = N \sum_m c_m \tanh^m(2|\xi|)|2m\rangle \tag{4.7}$$

and compute the coefficients c_m.

(d) The 'normal-ordered form' of the squeezing operator is given by (5 bonus points for a proof; maybe error in the phase)

$$\begin{aligned} S(\xi) &= \exp\left(\frac{\nu}{2\mu}a^{\dagger 2}\right)\exp\left(-(a^\dagger a + \tfrac{1}{2})\log\mu\right)\exp\left(-\frac{\nu^*}{2\mu}a^2\right), && (4.8)\\ \nu &= \mathrm{e}^{\mathrm{i}\arg\xi}\sinh(2|\xi|) && (4.9)\\ \mu &= \cosh(2|\xi|) && (4.10) \end{aligned}$$

Starting from this identity, compute the Q-function for the squeezed state $|\xi\rangle$, defined by

$$Q_\xi(\alpha) = \pi^{-1}\left|\langle\alpha|\xi\rangle\right|^2 \tag{4.11}$$

where $|\alpha\rangle$ is the usual coherent state. Your result gives a geometric interpretation of the "squeezed vacuum" described by the state $|\xi\rangle$.

Einführung in die Quantenoptik Wintersemester 2007/08

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Problem set 5 Hand out: 07 Dec 2007

Hand in: 21 Dec 2007

Problem 5.1 – Coherence and spectrum (6 points)

The (first order) coherence function of a field is defined by

$$g^{(1)}(t, t') = \langle E^{(-)}(t) E^{(+)}(t') \rangle. \tag{5.1}$$

The field is *stationary* if $g^{(1)}$ only depends on the time difference $t - t'$. It is *first-order coherent* if $g^{(1)}(t, t')$ factorizes: this means that a function $\mathcal{E}(t)$ exists such that $g^{(1)}(t, t') = \mathcal{E}^*(t)\mathcal{E}(t')$. The *spectrum* of a stationary field is defined by the Fourier transform of $g^{(1)}(t - t')$ with respect to $t - t'$.

(a) Show that a single-mode field is always first-order coherent.

(b) Show that a stationary, first-order coherent field is always monochromatic.

Problem 5.2 – Intensity correlations (6 points)

A particular version of the second-order coherence function is defined as

$$g^{(2)}(t, t') = \langle E^{(-)}(t') E^{(-)}(t) E^{(+)}(t) E^{(+)}(t') \rangle, \tag{5.2}$$

which is a normally ordered expectation value of field operators. It is measurable from a two-photon coincidence signal. The intensity correlation function is defined by

$$I^{(2)}(t, t') = \langle I(t) I(t') \rangle \tag{5.3}$$

where the intensity operator is $I(t) = E^{(-)}(t)E^{(+)}(t)$. This is measurable from the time series that a photodetector is recording.

(a) For a single-mode field, compute a link between the two quantities. Try to find an interpretation by working out the example of a one- and two-photon state.

(b) Compute $g^{(2)}(t, t)$ for a single-mode field in a number state, in a coherent state, in a thermal state, and in a squeezed state.

Problem 5.3 – Electric dipole coupling (8 points)

Consider an atomic system, small compared to the relevant wavelengths of the electromagnetic field, that is described by the approximate minimal coupling Hamiltonian

$$H_A = \sum_\alpha \frac{(\mathbf{p}_\alpha - q_\alpha \mathbf{A}(\mathbf{R}, t))^2}{2M_\alpha} + \frac{1}{2} \sum_{\alpha \neq \beta} \frac{q_\alpha q_\beta}{4\pi\varepsilon_0 |\mathbf{r}_\alpha - \mathbf{r}_\beta|}.$$

The approximation consists of evaluating the vector potential at a fixed reference point $\mathbf{R}$ at the center of the atom. The scalar potential is assumed to be zero. Show that the following local phase transformation

$$\psi(\{\mathbf{r}_\alpha\}, t) \mapsto \tilde{\psi}(\{\mathbf{r}_\alpha\}, t) = \exp\left[-\frac{\mathrm{i}}{\hbar} \sum_\alpha q_\alpha \chi(\mathbf{r}_\alpha, t)\right] \psi(\{\mathbf{r}_\alpha\}, t), \tag{5.4}$$

with the gauge function

$$\chi(\mathbf{r}, t) = (\mathbf{r} - \mathbf{R}) \cdot \mathbf{A}(\mathbf{R}, t) \tag{5.5}$$

leads to a Hamiltonian for $\tilde{\psi}$ where the vector potential disappears and where the interaction with the field is given by the electric dipole ("$d \cdot E$") coupling. Is this change of gauge compatible with the Coulomb gauge condition?

Hint. The transformation is time-dependent, therefore an extra term appears in the 'new Hamiltonian' for the wave function $\tilde{\psi}$. To show that the vector potential changes according to a gauge transformation, consider the expression

$$(\mathbf{p}_\beta - q_\beta \mathbf{A}(\mathbf{R}, t)) \exp\left[\frac{\mathrm{i}}{\hbar} \sum_\alpha q_\alpha \chi(\mathbf{r}_\alpha, t)\right] = \exp\left[\frac{\mathrm{i}}{\hbar} \sum_\alpha q_\alpha \chi(\mathbf{r}_\alpha, t)\right] (\mathbf{p}_\beta - q_\beta \mathbf{A}'(\mathbf{R}, t))$$

where the difference between $\mathbf{A}'$ and $\mathbf{A}$ arises from the momentum operator $\mathbf{p}_\beta$, acting on the exponential. Compute the new vector potential $\mathbf{A}'$ for the gauge function (5.5).