Einführung in die Quantenoptik Wintersemester 2007/08

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Problem set 6 Hand out: 21 Dec 2007

Hand in: 11 Jan 2008

Problem 6.1 – P- and Q-function (10 points)

(a) In the lecture, we have computed the Q-function for a Fock (number) state:

$$Q_n(\alpha) = \frac{1}{\pi}|\langle\alpha|n\rangle|^2 = \exp(-|\alpha|^2)\frac{|\alpha|^{2n}}{\pi\, n!} \tag{6.1}$$

Using the relations derived in the lecture between the P- and Q-functions, show that the P-“function” for the Fock state is given by the highly singular expression [Sudarshan, *Phys Rev Lett* **10** (1963) 277; no guarantee for errors]

$$P_n(\alpha) = \frac{\mathrm{e}^{|\alpha|^2}}{n!}\frac{\partial^n}{\partial\alpha^n}\frac{\partial^n}{\partial\alpha^{*n}}\delta^{(2)}(\alpha) \tag{6.2}$$

(b) Starting from Eq.(6.2), show that the average photon number in a Fock state $|n\rangle$ is given by … n and that its variance vanishes.

Solution. The direction calculation with the help of a deconvolution is tricky because one ends up with an integral representation of derivatives of the δ-function, or with a binomial sum of derivatives:

$$P_n(\alpha) = \sum_{k=0}^{n}\binom{n}{k}\frac{1}{k!}\frac{\partial^k}{\partial^k\alpha}\frac{\partial^k}{\partial^k\alpha^*}\delta^{(2)}(\alpha) \tag{6.3}$$

A more direct proof of Eq.(6.2) can be found in the book by Mandel & Wolf: observe that

$$\langle-\beta|\rho|\beta\rangle = \int \mathrm{d}^2\alpha\, P(\alpha)\langle-\beta|\alpha\rangle\langle\alpha|\beta\rangle = \mathrm{e}^{-|\beta|^2}\int \mathrm{d}^2\alpha\, P(\alpha)\,\mathrm{e}^{-|\alpha|^2}\mathrm{e}^{\beta\alpha^*-\beta^*\alpha} \tag{6.4}$$

where the Fourier transform of the product $P(\alpha)\,\mathrm{e}^{-|\alpha|^2}$ appears. The (formal) Fourier inversion then yields the representation

$$P(\alpha) = \mathrm{e}^{|\alpha|^2}\int\frac{\mathrm{d}^2\beta}{\pi^2}\,\langle-\beta|\rho|\beta\rangle\,\mathrm{e}^{|\beta|^2}\mathrm{e}^{-\beta\alpha^*+\beta^*\alpha} \tag{6.5}$$

which has, of course, very limited convergence properties as an ordinary function. Evaluating the matrix element $\langle-\beta|\rho|\beta\rangle$ for a Fock state is straightforward, and a formal calculation gives Eq.(6.2).

Problem 6.2 – Coherence of sunlight (10 points)

This is a problem to think about for the end of the year. The simplest model for the sunlight that reaches the earth is a multi-mode field that is made up from uncorrelated plane waves with some angular spread.

(a) Question: what is the spatial coherence function of sunlight, $g^{(1)}(\mathbf{r}, \mathbf{r}')$, after monochromatic filtering? You can use

$$\langle a^\dagger_{\mathbf{k}} a_{\mathbf{k}'} \rangle = \bar{n}(\omega_{\mathbf{k}}) \delta_{\mathbf{k}\mathbf{k}'} F(\hat{\mathbf{k}}) \tag{6.6}$$

where $F(\hat{\mathbf{k}})$ selects the relevant range of directions (of the order of $0.5°$ around the mean direction to the sun). To implement the frequency filter, include a δ-function fixing the mode frequencies to $\omega_{\mathbf{k}} = \omega$ under the k-summations/integrals.

(b) Determine the range of distances $\mathbf{r} - \mathbf{r}'$ where the coherence function $g^{(1)}(\mathbf{r}, \mathbf{r}')$ is not yet zero. (It decays to zero for large distances.) If you place two slits at $\mathbf{r}$, $\mathbf{r}'$ where $|g^{(1)}(\mathbf{r}, \mathbf{r}')| \neq 0$, you should see Young's interference fringes (even without a 'coherence slit'). The corresponding coherence distance is surprisingly large.

(c) What value would you guess for the coherence distance, before embarking on the calculation?

Hint. For simplicity, take a screen perpendicular to the average direction towards the sun and consider only points on the screen. Ignore polarization, the refractive effect of the atmosphere, scattering by clouds etc.