Einführung in die Quantenoptik Wintersemester 2007/08

Carsten Henkel

Problem set 2

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Problem 2.1 – Transverse δ-function (7 points)

In the lecture, we have encountered the so-called 'transverse δ-function $\delta^{\perp}_{ij}(\mathbf{x})$. It has the following properties:

$$F_i^{\perp}(\mathbf{x}) := \int \mathrm{d}^3x' \delta^{\perp}_{ij}(\mathbf{x}-\mathbf{x}')F_j(\mathbf{x}') \quad \text{is a transverse vector field} \tag{2.1}$$

$$\int \mathrm{d}^3x' \delta^{\perp}_{ij}(\mathbf{x}-\mathbf{x}')F_j(\mathbf{x}') = \begin{cases} F_i(\mathbf{x}) & \text{if } \mathbf{F}(\mathbf{x}) \text{ is transverse} \\ 0 & \text{if } \mathbf{F}(\mathbf{x}) = \nabla\phi(\mathbf{x}) \end{cases} \tag{2.2}$$

$$\delta^{\perp}_{ij}(\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3} \,\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} (\delta_{ij} - \frac{k_i k_j}{k^2}) \tag{2.3}$$

$$\delta^{\perp}_{ij}(\mathbf{x}) = \delta_{ij}\delta(\mathbf{x}) + \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{|\mathbf{x}|} \tag{2.4}$$

(1) Describe in words the meaning of (2.3).

(2) Show that Eq.(2.3) correctly implements property (2.2) by working with the spatial Fourier transform of the vector function $\mathbf{F}(\mathbf{x})$.

(3) Prove Eq.(2.4) by starting from Eq.(2.3).

(4) Find a second proof of Eq.(2.4) by analogy to a gauge transformation: Be $\mathbf{F}(\mathbf{x})$ any (sufficiently smooth) vector field. Consider it as a vector potential and find a gauge transformation such that the transformed $\mathbf{F}'(\mathbf{x})$ is transverse.

Hint. Observe that the 'gauge function' solves an inhomogeneous Laplace equation.

Problem 2.2 – Field quantization in a simple geometry (8 points)

Consider a cavity bounded by two parallel mirrors where the mode functions for the electromagnetic field take the form

$$\mathbf{f}_{\kappa}(\mathbf{x}) = N\mathbf{e}f(x) \tag{2.5}$$

where N is a normalization factor, x the coordinate perpendicular to the mirrors and $\mathbf{e}$ a spatially constant polarization vector.

(1) Find $\mathbf{e}$ such the mode function is transverse.

(2) Assume periodic boundary conditions on the interval $0 \le x \le L$ and construct a real-valued $f(x)$ that solves the Helmholtz equation.* Find N and an integration domain for the y, z coordinates such that the mode function is normalized as in the lecture.

* The Helmholtz equation is

$$\nabla^2 f + \frac{\omega^2}{c^2} f = 0 \tag{2.6}$$

(3) What is the eigenfrequency ω_κ of this mode? What changes if you adopt boundary conditions for a cavity with perfectly reflecting walls at $x = 0, L$?

(4) Continue with periodic boundary conditions and real-valued modes and bring the total field momentum

$$\mathbf{P} = \varepsilon_0 \int \mathrm{d}^3x\, \mathbf{E} \times \mathbf{B} \tag{2.7}$$

in the form of a sum over mode indices κ and annihilation and creation operators a_κ, $a_\kappa^\dagger$. You can use the mode expansion from the lecture for the vector potential to compute the fields. Try to give an interpretation of the result that the momentum density is not a simple sum over modes.

Solution. We discussed in the exercise session the special case of the field momentum for cavity with perfectly reflecting walls [part (3)]. The expansion for the electric field can be obtained taking the time derivative of the vector potential, and reads

$$\mathbf{E}(x) = \mathrm{i}\mathbf{e} \sum_k \sqrt{\frac{\hbar\omega_k}{\varepsilon_0 A L}} (a_k - a_k^\dagger) \sin(kx) \tag{2.8}$$

where the discrete k-vectors take the values $k = \pi/L, 2\pi/L, \ldots$. The magnetic field is given by

$$\mathbf{B}(x) = \mathbf{e}_x \times \mathbf{e} \sum_{k'} \sqrt{\frac{\hbar}{\varepsilon_0 \omega_{k'} A L}} (a_{k'} + a_{k'}^\dagger) k' \cos(k'x) \tag{2.9}$$

The field momentum is actually not correctly defined in Eq.(2.7): the operators $\mathbf{E}$ and $\mathbf{B}$ do not commute, and $\mathbf{P}$ would not be a hermitean operator. We symmetrize the product to get a hermitean momentum. To proceed, we need the following results

$$\mathbf{e} \times (\mathbf{e}_x \times \mathbf{e}) = \mathbf{e}_x$$

$$(a_k - a_k^\dagger)(a_{k'} + a_{k'}^\dagger) + (a_{k'} + a_{k'}^\dagger)(a_k - a_k^\dagger) = \text{symmetric under } k \leftrightarrow k' \tag{2.10}$$

$$\int_0^L \mathrm{d}x\, \sin(kx) k' \cos(k'x) = \int_0^L \mathrm{d}x\, \sin(kx) \frac{\partial}{\partial x} \sin(k'x)$$

$$= -\int_0^L \mathrm{d}x\, k \cos(kx) \sin(k'x) \tag{2.11}$$

Eq.(2.11) is found by partial integration. The integrated terms involve $\sin(kx)\sin(k'x)$ on the mirrors $x = 0, L$ and vanish there. Hence, this part is anti-symmetric under the $k \leftrightarrow k'$. (The special case $k = k'$ hence gives zero.)

Finally, we observe that the sum over k and k' involves a product of symmetric and anti-symmetric functions of k and k'. Hence, we get zero. The same result is obtained with periodic boundary conditions ($k = 0, 2\pi/L, 4\pi/L, \ldots$) and real mode functions $\sin(kx)$ and $\cos(kx)$.

It may still be asked how one can ever get a nonzero field momentum in a perfectly reflecting cavity. A physical situation where one would expect a nonzero momentum, at least temporarily, is a 'Q-switched laser' where a wavepacket (created by a superposition of modes) circulated in the cavity. This wavepacket may well be 'on its way to the right mirror' in the middle of the cavity and carry momentum like a laser beam or any plane plane wave does.

Problem 2.3 – The electric field per photon (5 points)

In the lecture, we have found the following mode expansion for the (transverse) vector potential (operator) in a 'quantization box' of volume V

$$\mathbf{A}(\mathbf{x}, t) = \sum_{\kappa} \sqrt{\frac{\hbar}{2\varepsilon_0 \omega_\kappa V}} \left(\mathbf{e}_\kappa \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega_\kappa t)} a_\kappa + \text{h.c.} \right). \tag{2.12}$$

where the label κ combines the wavevector and polarization of a given mode, $\mathbf{e}_\kappa$ is the polarization vector, and ω_κ is the mode frequency.

(1) Derive the electric field operator.

(2) In the following, we use this expansion as a rough estimate for the quantized field in a cavity. Take a quantization box with length L (distance between mirrors) and transverse area A (size of a focused beam). Focus on one mode and give an estimate for the 'electric field per photon' at the cavity centre. (I.e.: the amplitude of the mode function that multiplies the operator a_k or $a_k^\dagger$.)

(3) Take an atom of the size of the Hydrogen atom and estimate the electric dipole interaction energy V_{dip} for a mode nearly resonant with a typical transition between Hydrogen bound states. You can use that the electric dipole moment d is of the order of atomic size times electron charge. Compare this interaction energy to (i) the binding energy of Hydrogen and to (ii) the natural linewidth $\hbar\gamma = d^2 k^3 / 3\pi\varepsilon_0$ of a transition with wavenumber k. In the case that $V_{\text{dip}} > \hbar\gamma$, one says that a single photon can 'saturate' this transition. What cavity or atomic parameters are needed to enter this regime?