

1.5 Light detectors and coherence

1.5.1 Glauber photodetector theory

We now have to look in more detail at how exactly the quantum state of the electromagnetic field can be measured. The devices that measure light are called “photodetectors”. We shall sketch here the theory that R. Glauber developed to describe these detectors. Light detection has some particular properties: for example, it is practically impossible to resolve the oscillations of the electric field at an optical period of about 10^{-15} s — detectors are simply “too slow” for that. They have a “response time” that is typically in the 10^{-9} s range (“fast” photodetector). In practice (and also in the classical theory), one therefore has only access to time-averaged quantities, the average being taken over at least one optical period.

A good “quantum” model for a light detector is a two-level atom initially in its ground state that gets excited when it is illuminated by light. If we can measure the population of its excited state, we have a signal which is related to the light field. This is the starting point of Glauber’s theory.

Detector model. The details are the following: an efficient detection is possible when the two-level system is ionized by the light because the ejected electron can efficiently be detected (this is used in photomultipliers). This means, however, that our excited state is actually a continuum (the electron moves freely), and we cannot use a simple-minded two-state model. So let us assume the level scheme shown in figure 1.4 with a ground state $|g\rangle$ and a continuum C of excited states $|e\rangle$.

We are interested in the probability $p_C(t)$ that this detector goes into any of the excited states $|e\rangle$ (“detection probability”). In terms of the probability amplitudes $c_{e,g}(t)$, the detection probability is given by

$$p_C(t) = \sum_{e \in C} |c_e(t)|^2. \quad (1.178)$$

We start at $t = 0$ with the detector in the ground state. To compute the $c_e(t)$, nothing better can we do than perturbation theory, as we did in chapter 1. This gives

$$c_e(t) = -\frac{i}{\hbar} \int_0^t d\tau \langle e; f | V(\tau) | g; i \rangle e^{i\omega_{eg}\tau} \quad (1.179)$$

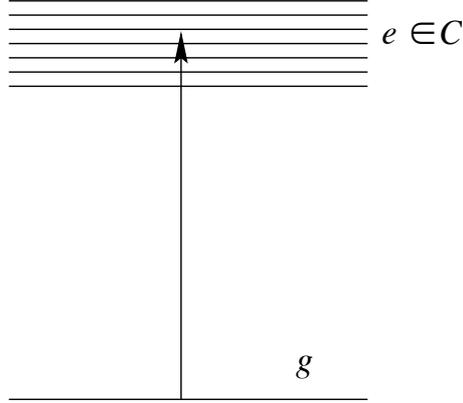


Figure 1.4: Two-level model for a photodetector.

where $|i\rangle$ and $|f\rangle$ are the initial and final field states. The interaction potential is taken in the electric dipole coupling

$$V(\tau) = -\hat{\mathbf{d}} \cdot \mathbf{E}(\tau) = -\hat{\mathbf{d}} \cdot \mathbf{E}^+(\tau) - \hat{\mathbf{d}} \cdot \mathbf{E}^-(\tau) \quad (1.180)$$

where $\mathbf{E}(\tau)$ is the electric field operator at the location of the detector, and $\mathbf{E}^\pm(\tau)$ its positive and negative frequency part: in the quantized description, the positive frequency part $\mathbf{E}^+(\tau)$ only contains annihilation operators $a_{\mathbf{k}}(\tau) \propto e^{-i\omega_{\mathbf{k}}\tau}$, and the negative frequency part only creation operators $a_{\mathbf{k}}^\dagger(\tau) \propto e^{i\omega_{\mathbf{k}}\tau}$.

Detection probability. Also in the quantum theory, we can make the resonance approximation (or equivalently, the r.w.a.) and keep in the time-integral (1.179) only those terms that lead to resonant denominators. Since the transition frequency ω_{eg} is positive, this amounts to keeping only the positive frequency part of the electric field operator. We thus have

$$c_e(t) = \frac{i}{\hbar} \mathbf{d} \cdot \int_0^t d\tau \langle f | \mathbf{E}^+(\tau) | i \rangle e^{i\omega_{eg}\tau} \quad (1.181)$$

where we have used the abbreviation $\mathbf{d} = \langle e | \hat{\mathbf{d}} | g \rangle$ and where the initial and final field states $|i, f\rangle$ have been spelled out. For the probability $p_C(t)$, we get an expression involving $\langle i | E_j^-(\tau') | f \rangle \langle f | E_i^+(\tau) | i \rangle$. At this point, we can perform the sum over the final field states $|f\rangle$ (these are not resolved, so

we add the corresponding probabilities). We thus get

$$p_C(t) = \frac{1}{\hbar^2} \sum_{e \in C} d^{j*} d^i \int_0^t d\tau d\tau' \langle i | E_j^-(\tau') E_i^+(\tau) | i \rangle e^{i\omega_{eg}(\tau-\tau')} \quad (1.182)$$

An explicit calculation of the sum over the continuum E is only possible with a specific model. Let us assume here the following form

$$\sum_{e \in C} d^{j*} d^i e^{i\omega_e \tau} = D \delta^{ij} \delta(\tau). \quad (1.183)$$

This form is plausible when the width $\Delta\omega_e$ of the continuum is sufficiently broad: for τ of the order of the time resolution of the detector, the phase factors $e^{i\omega_e \tau}$ then rapidly average out. Note that this behaviour is a consequence of the fact that “photodetectors are slow”. The Kronecker δ^{ij} for the field components is plausible when the detector does not discriminate the field polarization. This is a simplifying assumption that we relax later to describe polarization-sensitive detectors.

Putting (1.183) into the detection probability (1.182), we get the basic result

$$p_C(t) = \eta \int_0^t d\tau \langle \mathbf{E}^-(\tau) \cdot \mathbf{E}^+(\tau) \rangle, \quad \eta = \frac{D}{\hbar^2}, \quad (1.184)$$

where we see that a photodetector is sensitive to a “normally ordered” product of electric field operators (annihilation operators to the right and act first). The average $\langle \dots \rangle$ is an average over the field state — which can be a quantum average (in a pure quantum state $|i\rangle$) or a combined quantum and classical average. If the field is described by the density operator $\hat{\rho}$, then we obviously have

$$\langle \mathbf{E}^-(\tau) \cdot \mathbf{E}^+(\tau) \rangle = \text{tr} \left(\hat{\rho} \mathbf{E}^-(\tau) \cdot \mathbf{E}^+(\tau) \right).$$

The time-dependence of the operators reminds us that we deal with Heisenberg operators – the density matrix thus specifies the “initial” state (that does not evolve in time).

Photocurrent. The time derivative of the detection probability gives the production rate of photoelectrons, or (up to a factor e) the photocurrent $j(\mathbf{r}, t)$:

$$\begin{aligned} j(\mathbf{r}, t) &= \eta \langle I(\mathbf{r}, t) \rangle, \\ I(\mathbf{r}, t) &= : \mathbf{E}^-(\mathbf{r}, \tau) \cdot \mathbf{E}^+(\mathbf{r}, \tau) : \end{aligned} \quad (1.185)$$

The notation $: \dots :$ means “normal order” (creation operators to the left, annihilation operators to the right and act first). Up to this operator ordering prescription, we have found an expression that resembles the classical time-averaged Poynting vector (recall that it is also proportional to the squared complex field amplitude). Note that the average here is more involved than in the classical theory, since it also has quantum aspects.³ The proportionality factor η depends on the detector, and theoretical results are often normalized to make it disappear.

If the field is a single plane wave mode, we see that the photocurrent is proportional to the photon number operator:

$$j(t) = \eta_1 \langle a^\dagger(t) a(t) \rangle \quad (1.186)$$

with $\eta_1 = \eta E_{\text{1ph}}^2$. In this context, the factor η_1 is often interpreted as “detection efficiency”, i.e., the probability that a photon is detected.

Polarization filtering

Imagine a polarization filter in front of the photodetector. In classical terms, this means that only a component $\mathbf{u}^* \cdot \mathbf{E}$ is transmitted to the detector.⁴ In the quantum theory, we can describe this by expanding the field $\mathbf{E}(\mathbf{r}, t)$ into modes polarized parallel and perpendicular to \mathbf{u} . Only the mode with parallel polarization has a mode function that reaches the detector (recall that the calculation of mode functions is a classical problem, independent of the field quantization). So our photocurrent becomes

$$j_{\mathbf{u}}(\mathbf{r}, t) = \langle \mathbf{u} \cdot \mathbf{E}^-(\mathbf{r}, t) \mathbf{u}^* \cdot \mathbf{E}^+(\mathbf{r}, t) \rangle$$

Frequency spectrum

What happens with filtering in frequency space? We can in fact use a very similar reasoning: a classical mode at frequency ω will be transmitted to

³A more precise theory is possible at the classical level (“optical coherence theory”): there, the light field is treated as a stochastic field with fluctuations, since it often originates from many uncorrelated sources (classical representation of individual atoms). In the theory, an extra average over these fluctuations then enters. Experimentally, this average enters because detectors are “slow” and cannot resolve these fluctuations.

⁴The complex conjugate \mathbf{u}^* is applicable when a filter for circular polarization is used.

the detector with an efficiency given by the transmission coefficient of the filter. This quantity can be calculated for a cavity, e.g., and shows peaks (maximum transmission) at specific resonance frequencies. Let us focus on one of these frequencies, say ω_c , and suppose that the filter has a very narrow resonance. The transmitted field is then proportional to the time-Fourier transformed electric field at the resonance frequency ω_c :

$$\mathbf{E}_{\text{tr}}^+(t) \propto \tilde{\mathbf{E}}^+(\omega_c) e^{-i\omega_c t}$$

The corresponding photocurrent is then given by

$$\begin{aligned} j_{\omega_c}(t) &\propto \langle \tilde{\mathbf{E}}^-(\omega_c) \cdot \tilde{\mathbf{E}}^+(\omega_c) \rangle \\ &= \int dt_1 dt_2 e^{-i\omega_c(t_1-t_2)} \langle \mathbf{E}^-(t_1) \cdot \mathbf{E}^+(t_2) \rangle \end{aligned} \quad (1.187)$$

If we assume that the field is stationary in time, we can shift the time arguments in the product of field averages and write

$$j_{\omega_c}(t) \propto T \int d\tau e^{i\omega_c \tau} \langle \mathbf{E}^-(0) \cdot \mathbf{E}^+(\tau) \rangle \quad (1.188)$$

where T is the measurement time (at least equal to the time needed to build up the field in the cavity). This equation can be generalized to any detector (“spectrometer”) that measures a frequency spectrum. Finally, if the field changes slowly (on the timescale of the measurement time T), we can write

$$j_{\omega_c}(t) \propto T \int d\tau e^{i\omega_c \tau} \langle \mathbf{E}^-(t - \tau/2) \cdot \mathbf{E}^+(t + \tau/2) \rangle.$$

This quantity is called the ‘Wigner transform’ of $\langle \mathbf{E}^-(t) \cdot \mathbf{E}^+(t') \rangle$, a function that combines information in real time and in frequency space.

1.5.2 Coherence theory

We thus learn that with a spectrometer, we can extract the Fourier transform (with respect to τ) of the *field correlation function*

$$G^{(1)}(t, \tau) = \langle \mathbf{E}^-(t) \cdot \mathbf{E}^+(t + \tau) \rangle \quad (1.189)$$

This correlation function (also called “first order coherence function”) plays an important role when one wants to characterize the “coherence” of light.

One uses in practice often its normalized form

$$g^{(1)}(t, \tau) = \frac{\langle \mathbf{E}^-(t) \cdot \mathbf{E}^+(t + \tau) \rangle}{\langle \mathbf{E}^-(t) \cdot \mathbf{E}^+(t) \rangle^{1/2} \langle \mathbf{E}^-(t + \tau) \cdot \mathbf{E}^+(t + \tau) \rangle^{1/2}} \quad (1.190)$$

By definition, $g^{(1)}(t, 0) = 1$. The behaviour for large $|\tau|$ depends on the frequency spectrum of the field: if it is broad, then the different Fourier components rapidly get out of phase, and $|g^{(1)}(t)|$ decreases. For a stationary field, it can be shown using the Schwarz inequality that

$$|g^{(1)}(t, \tau)| \leq g^{(1)}(t, 0) = 1. \quad (1.191)$$

When equality holds, one says that the field is (first order) ‘coherent’. One can show that this condition is equivalent to the ‘factorization’ property:

$$\langle \mathbf{E}^-(t_1) \cdot \mathbf{E}^+(t_2) \rangle = \mathcal{E}^*(t_1) \cdot \mathcal{E}(t_2) \quad (1.192)$$

where $\mathcal{E}(t)$ is some complex function (not necessarily equal to the average field).

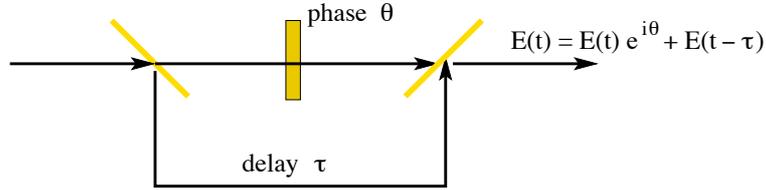


Figure 1.5: Setup to measure directly the temporal coherence function.

Direct measurement of temporal coherence. In the setup sketched in Fig.1.5, a field is split and one part delayed by some time τ . With a phase shifting plate, the phase difference of the recombining fields is controlled. A photodetector measures at the output an intensity

$$\begin{aligned} j_{\text{out}}(t) &= \eta \langle [E^{(-)}(t)e^{-i\theta} + E^{(-)}(t - \tau)][E^{(+)}(t)e^{i\theta} + E^{(+)}(t - \tau)] \rangle \\ &= \eta \langle E^{(-)}(t)E^{(+)}(t) \rangle + \langle E^{(-)}(t - \tau)E^{(+)}(t - \tau) \rangle \\ &\quad + 2 \left[e^{-i\theta} \langle E^{(-)}(t)E^{(+)}(t - \tau) \rangle + \text{c.c.} \right] \\ &= \eta G^{(1)}(t, t) + \eta G^{(1)}(t - \tau, t - \tau) \\ &\quad + 2\eta \left[e^{-i\theta} G^{(1)}(t, t - \tau) + \text{c.c.} \right] \end{aligned} \quad (1.193)$$

The mixed term directly measures the coherence function. It is the only one that depends on the relative phase θ . By plotting the detector current vs. θ , one gets sinusoidal ‘interference fringes’ with a contrast given by

$$\frac{j_{\max} - j_{\min}}{j_{\max} + j_{\min}} = \frac{2|G^{(1)}(t, t - \tau)|}{G^{(1)}(t, t) + G^{(1)}(t - \tau, t - \tau)}. \quad (1.194)$$

The field is called ‘coherent’ when this contrast is 100%. This condition is equivalent to $|G^{(1)}(t, t - \tau)| = [G^{(1)}(t, t)G^{(1)}(t - \tau, t - \tau)]^{1/2}$, which corresponds to first-order coherence, see after Eq.(1.191). We have by this calculation also demonstrated the inequality (1.191): indeed, if $|g^{(1)}(t, t - \tau)|$ were larger than 1, one would get a negative value for j_{\min} . *The temporal coherence function is thus a direct measure to what extent the field is ‘able to interfere’ with itself.*

As an exercise, you are asked to compute the first order coherence function for a single mode in a number state, a coherent state, and in a thermal state. You see there that single-mode fields are always first order coherent, whatever their state. This is because the time-dependence reduces to a single exponential $e^{-i\omega\tau}$ whose magnitude never decreases below unity. Only multi-mode fields with different mode frequencies can be first-order incoherent – their phase factors $e^{-i\omega\tau}$ dephase when τ gets larger than the typical frequency spacing. As an example, we consider the full electromagnetic field at thermal equilibrium.

Coherence function of thermal field. Armed with the informations on product states and multi-mode fields given in the previous digression, we can compute the first order coherence function of a multi-mode field in a thermal state. This function is defined by

$$G^{(1)}(\mathbf{x}, t, \tau) = \langle \mathbf{E}^-(\mathbf{x}, t) \cdot \mathbf{E}^+(\mathbf{x}, t + \tau) \rangle$$

where $\mathbf{E}^\pm(\mathbf{x}, t)$ are the positive (negative) frequency parts of the field. Using the mode expansion, we get

$$G^{(1)}(\mathbf{x}, t, \tau) = \sum_{\mathbf{k}, \mu} \sum_{\mathbf{k}', \mu'} E_{\mathbf{k}} E_{\mathbf{k}'} \boldsymbol{\varepsilon}_{\mathbf{k}\mu}^* \cdot \boldsymbol{\varepsilon}_{\mathbf{k}'\mu'} \langle a_{\mathbf{k}\mu}^\dagger a_{\mathbf{k}'\mu'} \rangle e^{i\omega_{\mathbf{k}}t} e^{-i\omega_{\mathbf{k}'}(t+\tau)} e^{i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{x}}$$

Now, we can easily generalize the result obtained before for the two-operator average and write

$$\langle a_{\mathbf{k}\mu}^\dagger a_{\mathbf{k}'\mu'} \rangle = \delta_{\mathbf{k}, \mathbf{k}'} \delta_{\mu, \mu'} \bar{n}(\omega_{\mathbf{k}}, T)$$

where again the average photon number appears. We have to specify the dependence on the frequency ω_k . With the Kronecker δ 's, we can collapse the sum to

$$\begin{aligned} G^{(1)}(\mathbf{x}, t, \tau) &= \sum_{\mathbf{k}, \mu} E_k^2 \bar{n}(\omega_k, T) e^{-i\omega_k \tau} \\ &= 2 \frac{V}{(2\pi)^3} \int d^3k \frac{\hbar \omega_k}{2\epsilon_0 V} \bar{n}(\omega_k, T) e^{-i\omega_k \tau} \end{aligned}$$

The factor 2 comes from the sum over the polarizations. The integral over the angles of \mathbf{k} gives 4π , and the radial integral can be written as a frequency integral, using $\omega = kc$. This gives

$$G^{(1)}(\mathbf{x}, t, \tau) = \frac{\hbar}{\epsilon_0 c^3} \frac{8\pi}{(2\pi)^3} \int d\omega \omega^3 \bar{n}(\omega, T) e^{-i\omega \tau}$$

Due to the exponential decay of the average photon number, the integrand has a frequency width of the order of $\Delta\omega \sim k_B T / \hbar$. This translates, via Fourier transformation, into a width $\tau_c \sim 1/\Delta\omega = \hbar/k_B T$. At 300 K, we get $\tau_c \sim 10^{-12} \text{ s}^{-1}$ – a very short time. For $\tau \ll \tau_c$, the integral is close to its value $\pi^4 \Delta\omega^4 / 15$, while in the opposite limit, $\tau \gg \tau_c$, it decays rapidly to zero. For typical measurement times, the thermal electromagnetic field is thus ‘incoherent’.

The normalized first-order coherence function $g^{(1)}(\tau)$ is plotted in fig. 1.6 and shows indeed a temporal width of the order of τ_c . It is by the way a general rule that the width τ_c of the first order coherence function is given by the inverse width of the frequency spectrum of the light. This is in agreement with the heuristic ‘dephasing rule’ given above.

If a field has a short coherence time τ_c , the setup sketched in Fig.1.5 is able to measure delays with a precision comparable to τ_c : one just checks whether interference fringes are seen. This situation occurs when the light fed into the device has a very broad spectrum (‘white light interferometry’). The group of R. Menzel (Universität Potsdam) performs experiments in this direction. The broad spectrum is created starting from a powerful laser beam with nonlinear effects in a fibre.

Example: squeezing detection. Combining a photodetector with a beamsplitter, one can detect the squeezing of a ‘signal beam’. Consider

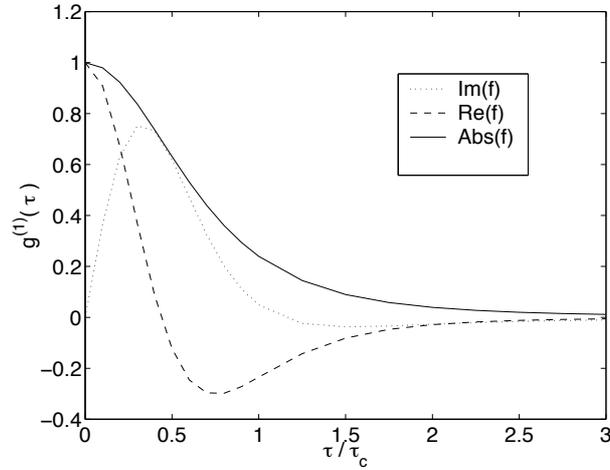


Figure 1.6: First-order temporal coherence function of the thermal electromagnetic field. Time is scaled to the coherence time $\tau_c = \hbar/k_B T$.

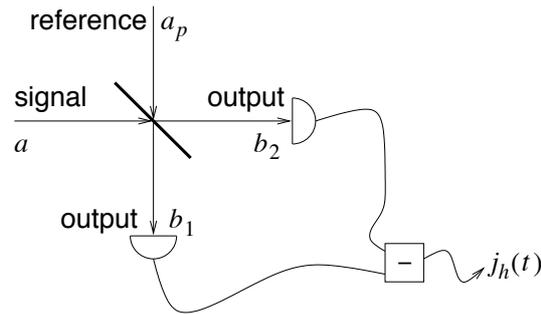


Figure 1.7: Setup for detecting squeezing.

the experiment schematically shown in Fig. 1.7. A signal beam and a reference beam are “mixed” at a beamsplitter, and the two output beams are detected by photodetectors. The difference in photocurrent, $j_h(t)$, and its fluctuations are the observables. This detection scheme is called “*homodyne detection*”.

You have seen in the exercises that the output mode operators b_1, b_2 after the beamsplitter are related to the input mode operators a, a_p via the “beamsplitter transformation”:

$$\begin{aligned} b_1 &= Ra + Ta_p \\ b_2 &= T^*a - R^*a_p \end{aligned} \tag{1.195}$$

The photocurrents in the two detectors are the expectation values of the operators $b_1^\dagger b_1$ and $b_2^\dagger b_2$. For detector 1, e.g., we get

$$b_1^\dagger b_1 = |T|^2 a_p^\dagger a_p + R^* T a^\dagger a_p + R T^* a_p^\dagger a + |R|^2 a^\dagger a.$$

In the limit of a reference beam in a coherent state with a large amplitude α_p , the first two terms in this result are dominant. The first is actually a constant offset $|T\alpha_p|^2$ that can be subtracted in the experiment. The second term now gives access to a particular field quadrature: writing $R^* T \alpha_p = C e^{-i\theta}/\sqrt{2}$, we find

$$b_1^\dagger b_1 = \text{const.} + C X_\theta$$

where the quadrature X_θ was defined in (1.130). Homodyne measurements that mix the signal with a strong reference field thus enable us to measure also arbitrary quadratures. The phase θ can be scanned by scanning the phase of the reference beam. Recall that with a simple photodetection measurement of the beam a , we have only access to the photon number $a^\dagger a$.

The subtraction of the constant current due to the reference beam can be done in a more elegant way by measuring the photocurrent difference, as shown in fig. 1.7. Using the beamsplitter transformation (1.195), the current difference is the average of the operator

$$b_1^\dagger b_1 - b_2^\dagger b_2 = (|T|^2 - |R|^2) a_p^\dagger a_p - (|R|^2 - |T|^2) a^\dagger a + 2C X_\theta$$

with the same constants C and θ . The first and second terms involving simple photon number operators can be cancelled by choosing a 50/50 beamsplitter with $|R|^2 = |T|^2 = \frac{1}{2}$ ('balanced homodyne detection'). The difference current is then directly given by

$$j_h(t) = 2C \langle X_\theta \rangle$$

Note that in the present case, the current is time-dependent when the signal field is not stationary.

Consider now the "noise" of the difference current. This corresponds to the observable

$$(\Delta j_h(t))^2 = 4C^2 (\Delta X_\theta)^2.$$

It is now clear that one can detect the squeezing of the signal beam: whenever the noise $(\Delta X_\theta)^2$ for a quadrature angle θ drops below the standard

quantum limit $\frac{1}{2}$, this quadrature is “squeezed”. It is interesting to note that it is sufficient to block the signal beam to get the reference noise level “ $\frac{1}{2}$ ”. The signal beam is then in the vacuum state, and one has $\langle X_\theta \rangle = 0$ and $(\Delta X_\theta)^2 = \frac{1}{2}$ for any reference phase θ . This noise level is commonly called “shot noise” and interpreted in terms of the independent photons detected out of the coherent state of the reference beam. The amazing thing is that when the vacuum signal beam is replaced by a squeezed vacuum, then still $\langle X_\theta \rangle = 0$, so there is no average detector signal, but the detection noise decreases, $(\Delta X_\theta)^2 < \frac{1}{2}$, for some choices of the reference phase. In the exercises, you are asked to study this process in more detail.

Heterodyne measurements. There exists a variant of the homodyne detection where signal and reference beam do not have the same frequency. The difference photocurrent is then time-dependent and shows a beat note (*Schwebung*) at the difference frequency $\omega_{\text{ref}} - \omega$. The current correlation function $\langle j_h(t + \tau)j_h(t) \rangle$ also shows an oscillation as a function of τ at the difference frequency. This can be detected with a “spectrum analyzer” that computes the Fourier transform of the current correlation function. Detection at a nonzero frequency has the advantage that residual, low-frequency noise is reduced and the signal-to-noise ratio is improved.

1.5.3 Intensity correlations

The expression (1.185) only takes into account the absorption of single photons (“one by one”) in the detector. But it is also possible to detect more than one photon. In particular, one can measure coincidences, i.e., the number of events where one photon is detected at time t and another one at time $t + \tau$. How is this quantity (an *intensity correlation function*) expressed in terms of field operators?

To answer this question, we extend the photodetector model to contain a large number of two-level atoms that can absorb a photon and be excited into a continuum of excited levels from which they eject electrons that we detect. We want to compute the joint probability that one atom (any one of the many) gets excited at time t and another one at time $t + \tau$. Since we deal with many atoms and assume that the excitation probability is small

(perturbation theory), it is very improbable that the same atom gets excited twice.

We go into the interaction representation and have the following Schrödinger equation for the state of the detector:

$$i\hbar\partial_t|\psi\rangle = H_{\text{int}}(t)|\psi\rangle$$

where the time-dependence of $H_{\text{int}}(t)$ includes the free evolution of the detector atoms, $H_{\text{int}}(t) = \exp(itH_0/\hbar) V(t) \exp(-itH_0/\hbar)$. The formal solution of this equation is

$$|\psi(t)\rangle = \mathcal{T}_+ \exp\left[-\frac{i}{\hbar} \int_0^t d\tau H_{\text{int}}(\tau)\right] |\psi(0)\rangle$$

where \mathcal{T}_+ means “positive time ordering”: in the series expansion of the exponential, order products of operators $H_{\text{int}}(\tau)$ such that their time-arguments increase from right to left.

We assume as before that at $t = 0$, all detector atoms are in the ground state and denote this state $|g\rangle$. The probability amplitude that two atoms get excited between 0 and t is given in lowest order perturbation theory by the second-order term of the expansion of the exponential. We order the product $H_{\text{int}}(t_1)H_{\text{int}}(t_2)$ in time such that $t_2 > t_1$. The corresponding state is

$$U^{(2)}(t)|g; i\rangle = -\frac{1}{\hbar^2} \int_0^t dt_1 dt_2 \Big|_{t_2 > t_1} H_{\text{int}}(t_2)H_{\text{int}}(t_1)|g; i\rangle$$

where the interaction Hamiltonian acts twice on the initial state. We now specify that our detector has N two-level atoms and that the interaction is a sum over all atoms a , $H_{\text{int}}(t) = \sum_a H_{\text{int}}^{(a)}(t)$, where $H_{\text{int}}^{(a)}(t)$ acts only in the Hilbert space for atom a (and on the field state, of course). After time t_1 , we have in the resonance approximation one excited atom:

$$H_{\text{int}}(t_1)|g\rangle = \sum_{a=1}^N |e_a; g\rangle \langle e_a| H_{\text{int}}^{(a)}(t_1)|g_a\rangle$$

where the state $|e_a; g\rangle$ is shorthand for “atom a excited and the others in the ground state”. The matrix element is taken in a one-atom Hilbert space (and still contains field operators) because the other atoms are unchanged. Similarly, after time t_2 , we have a state with two atoms excited:

$$H_{\text{int}}(t_2)|e_a; g\rangle = \sum_{b=1}^N |e_a; e_b; g\rangle \langle e_b| H_{\text{int}}^{(b)}(t_2)|g_b\rangle$$

With this interpretation, we can motivate the following expression for the probability amplitude that one photon is absorbed at time t and another one at time $t + \tau$, when the detector is switched on during short intervals Δt :

$$A_{ab}(t, t + \tau) = -\frac{1}{\hbar^2} \int_{t+\tau}^{t+\tau+\Delta t} dt_2 \int_t^{t+\Delta t} dt_1 \langle f | \langle e_b | H_{\text{int}}^{(b)}(t_2) | g_b \rangle \langle e_a | H_{\text{int}}^{(a)}(t_1) | g_a \rangle | i \rangle \quad (1.196)$$

We now proceed to the corresponding coincidence probability. The calculation is quite similar to that for single photon detection: take only the positive frequency part of the interaction Hamiltonian in matrix elements like

$$\langle e_a | H_{\text{int}}^{(a)}(t_1) | g_a \rangle = -\mathbf{d}_{eg} \cdot \mathbf{E}^+(t_1) e^{i\omega_{eg}t_1};$$

square the amplitude (1.196); sum over the final field state $|f\rangle$ and the continuum of excited atomic states. We also sum over all $N(N-1) \approx N^2$ ordered pairs of atoms a, b . Using the photodetector coefficient D defined in (1.184), the result is the probability

$$p_2(t, t + \tau) = \frac{D^2 N^2}{\hbar^4} \int_{t+\tau}^{t+\tau+\Delta t} dt_2 \int_t^{t+\Delta t} dt_1 \sum_{ij} \langle E_i^-(t_1) E_j^-(t_2) E_j^+(t_2) E_i^+(t_1) \rangle \quad (1.197)$$

where the i, j are the cartesian indices of the field operators. The ordering of the positive frequency operators corresponds to the ordering in eq. (1.196). The negative frequency operators have the reverse order because the probability involves the hermitian conjugate.

If the measurement time Δt is short compared to the time scale on which the field changes, the double time integral is proportional to Δt^2 . We then find a photon coincidence rate (for $\tau > 0$)

$$j_2(\mathbf{r}, t, t + \tau) = \eta^2 \sum_{ij} \langle E_i^-(\mathbf{r}, t) E_j^-(\mathbf{r}, t + \tau) E_j^+(\mathbf{r}, t + \tau) E_i^+(\mathbf{r}, t) \rangle \quad (1.198)$$

with $\eta = DN/\hbar^2$ the single photon detector efficiency. We have restored the position dependence in eq. (1.198), assuming that all detector atoms essentially “see” the same electric field (the detector is small compared to the scale on which the field varies). This result can be generalized to correlations between detectors at different locations, as well as to more than two photon detections.

Second-order coherence. A field is called “second order coherent” when the intensity correlation function factorizes, similar to the condition we introduced in eq. (1.192) for first-order coherence.

$$\begin{aligned}
 G^{(2)}(t, \tau) &:= \sum_{ij} \langle E_i^-(\mathbf{r}, t) E_j^-(\mathbf{r}, t + \tau) E_j^+(\mathbf{r}, t + \tau) E_i^+(\mathbf{r}, t) \rangle \\
 &= \sum_{ij} \mathcal{E}_i^*(\mathbf{r}, t) \mathcal{E}_j^*(\mathbf{r}, t + \tau) \mathcal{E}_j(\mathbf{r}, t + \tau) \mathcal{E}_i(\mathbf{r}, t) \quad (1.199)
 \end{aligned}$$

Here $\mathcal{E}(t)$ is the complex function appearing in the first-order coherence condition (1.192).

In the exercises, you are asked to compute the second-order coherence function for a single-mode field in a number state and in a coherent state. You find that a number state is only first order coherent: indeed, the first and second order coherence functions (at $\tau = 0$) are proportional to n and $n(n - 1)$, while the factorization property predicts a result proportional to n^2 for $G^{(2)}(t, 0)$. The coherent state, however is also second order coherent. (In fact it is even coherent to all orders because it is an eigenstate of the annihilation operator appearing in $E^+(t)$.)

1.6 Phase space functions

We have used in this chapter two pictorial representation of quantum states of a single radiation mode (a harmonic oscillator) in the phase space (actually the “phase plane”). This section aims at giving these pictures a better foundation. The idea is to find functions on the phase plane that characterize the state (or the density matrix) of a single mode. Coherent states $|\alpha\rangle$ are an essential ingredient for this programme because they are “localised” around phase space points (q, p) with $\alpha = q + ip$. We shall see that there are subtle differences with respect to ordinary expansions of the density operator; this is because the coherent states are not orthogonal (they cannot since they are eigenstates of a non-hermitean operator).

In this section, we focus on a single mode of the radiation field. The density operator thus acts on the Fock space spanned by the (orthogonal) number state basis $|0\rangle, |1\rangle, \dots$

1.6.1 Coherent states: a reminder

The coherent state $|\alpha\rangle$ is an eigenstate of the annihilation operator

$$a|\alpha\rangle = \alpha|\alpha\rangle$$

Its expansion in the number state basis reads

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle.$$

We use the standard phase convention that $\langle 0|\alpha\rangle$ is real and positive.

The coherent states are *complete* in the sense that one can write the unit operator as

$$\mathbb{1} = \int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| \quad (1.200)$$

where the integration measure is the phase space volument element

$$d^2\alpha = d(\operatorname{Re} \alpha) d(\operatorname{Im} \alpha) = dq dp = r dr d\varphi \quad (\alpha = r e^{i\varphi})$$

Proof: let us compute the matrix elements of this operator in the (complete) number state basis.

$$\int \frac{d^2\alpha}{\pi} \langle n|\alpha\rangle\langle\alpha|m\rangle = \int \frac{d^2\alpha}{\pi} e^{-|\alpha|^2} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}}$$

Change to radial $r = |\alpha|$ and angular coordinates $\varphi = \arg \alpha$:

$$= \int \frac{r dr d\varphi}{\pi} e^{-r^2} r^{n+m} \frac{e^{i(n-m)\varphi}}{\sqrt{n! m!}}$$

The angular integral gives $2\pi\delta_{nm}$. The radial integral is performed using the variable $x = r^2$:

$$= \int_0^\infty 2r dr \frac{e^{-r^2} r^{2n}}{n!} = \int_0^\infty dx \frac{e^{-x} x^n}{n!} = 1.$$

By a similar calculation, one can show that the coherent states are “over-complete”: they are not linearly independent and the following linear combination, for example, gives zero

$$n = 1, 2, \dots : \quad \int d^2\alpha |\alpha\rangle \alpha^n e^{-|\alpha|^2/2} = 0.$$

(Instead of the exponential, any function could be used that leads to a convergent radial integral.)

Finally, the coherent states are not orthogonal. In the number state basis, it is easy to show that

$$\langle \alpha | \beta \rangle = \exp[-\frac{1}{2}|\alpha - \beta|^2 + \frac{1}{2}(\alpha^* \beta - \alpha \beta^*)] \quad (1.201)$$

so that $|\langle \alpha | \beta \rangle|^2$ tends to zero if $|\alpha - \beta| \gg 1$. If α and β are “close”, the overlap is substantial. This justifies the finite extension of the “blobs” in the phase plane that we used to illustrate coherent states.

The formula (1.201) can also be obtained using the displacement operator $D(\alpha)$. Recall that the latter is defined by

$$|\alpha\rangle = D(\alpha)|0\rangle$$

and has the explicit form

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a} = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} e^{-\alpha^* a} \quad (1.202)$$

The second expression can be proven from the Baker-Hausdorff formula and gives the “normally ordered form” (annihilation operators act first). The same formula allows to prove the “composition law”

$$D(\alpha)D(\beta) = D(\alpha + \beta) \exp \frac{1}{2}(\alpha\beta^* - \alpha^*\beta) \quad (1.203)$$

where the exponential is a pure phase factor (and was missing in an earlier version of the script). Using $[D(\alpha)]^\dagger = D(-\alpha)$, one finds (1.201).

1.6.2 The P - and Q -functions

The P - and Q -functions⁵ are related to two naive ideas to form a distribution function for a density matrix in terms of coherent states. Since these states are not orthogonal, the two constructions do not give the same function. The Q -function is the expectation value of the density matrix in a coherent state:

$$Q(\alpha) = \frac{1}{\pi} \langle \alpha | \rho | \alpha \rangle \quad (1.204)$$

It is also called the Husimi function. The P -function (or Glauber-Sudarshan distribution) is the expansion coefficient when the density matrix is expanded in projectors on coherent states:

$$\rho = \int d^2\alpha P(\alpha) |\alpha\rangle \langle \alpha|. \quad (1.205)$$

Taking the expectation value of this definition in the coherent state $|\beta\rangle$, we find immediately the relation

$$Q(\beta) = \int \frac{d^2\alpha}{\pi} P(\alpha) e^{-|\alpha-\beta|^2} \quad (1.206)$$

showing that one gets the Q -function by “smearing out” the P -function over regions of radius 1 in the phase space. It follows that the P -function can have weird mathematical singularities on small scales in phase space.

Properties of the Q -function

From its definition, we see that it is no longer an operator, but a real and positive function. For a pure coherent state $|\beta\rangle$, it is a gaussian, as shown by (1.201):

$$Q_\beta(\alpha) = \frac{1}{\pi} |\langle \alpha | \beta \rangle|^2 = \frac{1}{\pi} \exp(-|\alpha - \beta|^2)$$

This justifies our pictures of coherent states as being “circular blobs” in phase space with radius 1. The factor $1/\pi$ ensures that the Q -function is correctly normalized (using the “resolution of the identity” (1.200))

$$\int d^2\alpha Q(\alpha) = \int \frac{d^2\alpha}{\pi} \text{tr}(|\alpha\rangle \langle \alpha| \rho) = \text{tr} \rho = 1$$

⁵No relation to quadrature or momentum and position variables, just historical notation!

For a number state, we get a “Poissonian” distribution (although here, n is a parameter and α the independent variable):

$$Q_n(\alpha) = \frac{1}{\pi} |\langle \alpha | n \rangle|^2 = \exp(-|\alpha|^2) \frac{|\alpha|^{2n}}{\pi n!}$$

For $n \gg 1$, the maximum of this distribution lies on the circle $|\alpha| \sim \sqrt{n}$. See the sketch in figure 1.8.

Finally, as an exercise, we quote the Q -function for a squeezed vacuum state $|\xi\rangle$ (with the squeezing parameter ξ). The squeezing operator can be normally ordered

$$|\xi\rangle = S(\xi)|0\rangle, \quad S(\xi) = \exp\left[-\frac{1}{2}(\xi a^{\dagger 2} - \xi^* a^2)\right] = e^{-\frac{\nu}{2\mu} a^{\dagger 2}} \left(\frac{1}{\mu}\right)^{\hat{n} + \frac{1}{2}} e^{\frac{\nu^*}{2\mu} a^2}$$

with $\mu = \cosh |\xi|$, $\nu = e^{i\varphi_\xi} \sinh |\xi|$. From this, one can show that the Q -function has an elliptic maximum centred at zero with radii $(1 \pm \tanh |\xi|)^{-1/2}$ along the quadrature axes that show maximum and minimum squeezing, respectively. This justifies the drawings of figure 1.8 where contour plots of the Q -function are sketched.

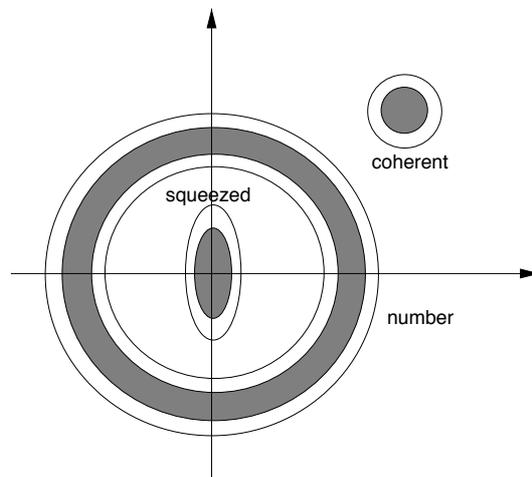


Figure 1.8: Contour plots of the Q -function for coherent, number and squeezed states.

What is the Q -function good for? One can “easily” compute expectation values of anti-normally ordered operators. For a proof, consider a typical

term of an anti-normally ordered operator, $A^{(a)}(a, a^\dagger) = a^n a^{\dagger m}$. Its expectation value is

$$\begin{aligned} \text{tr} (a^n a^{\dagger m} \rho) &= \int \frac{d^2\alpha}{\pi} \text{tr} (a^n |\alpha\rangle \langle \alpha| a^{\dagger m} \rho) \\ &= \int \frac{d^2\alpha}{\pi} \alpha^n \text{tr} (|\alpha\rangle \langle \alpha| \rho) \alpha^{*m} \\ &= \int \frac{d^2\alpha}{\pi} \alpha^n \alpha^{*m} \underbrace{\langle \alpha | \rho | \alpha \rangle}_{\pi Q(\alpha)} \end{aligned}$$

We conclude that the expectation of general anti-normally ordered operators can be computed like a classical average

$$\langle A^{(a)}(a, a^\dagger) \rangle = \int d^2\alpha A^{(a)}(\alpha, \alpha^*) Q(\alpha)$$

where the operators are replaced (in the most intuitive way) by c-numbers and the function $Q(\alpha)$ plays the role of a (classical) probability distribution. Vacuum fluctuations exist in this picture because even in the vacuum state $|0\rangle$ (a special coherent state), the Q -function has a nonzero extension around the origin of the phase space.

Properties of the P -function

It is no operator, but a real function (because the density matrix is hermitean). That is all we can say for the moment.

For a coherent state $|\beta\rangle$, it is a δ -function

$$P_\beta(\alpha) = \delta^{(2)}(\alpha - \beta)$$

where $\delta^{(2)}$ is a real function defined as one would expect for integration over α :

$$\delta^{(2)}(\alpha) = \delta(\text{Re } \alpha) \delta(\text{Im } \alpha).$$

This already shows that one easily faces P -“distributions” rather than “functions”.

It gets worse for number states $|n\rangle$: one can check that in this case, the P -“function” contains derivatives of a $\delta^{(2)}$ -function up to n th order:

$$P_n(\alpha) = \sum_{k=0}^n \frac{1}{k!} \binom{n}{k} \frac{\partial^k}{\partial \alpha^k} \frac{\partial^k}{\partial \alpha^{*k}} \delta^{(2)}(\alpha)$$

We recall that the P -function only gives positive, finite results when it is smeared out over a phase space area of order unity (see the relation (1.206) to the Q -function).

How can the P -function calculated from the density operator? This was easy for the Q -function (simply take the expectation value), but it is more complicated here. One can either start from the Q -function and invert the “smearing out” relation (1.206). Alternatively, one can use (quoted here without proof)

$$P(\alpha) = \int \frac{d^2\beta}{\pi^2} e^{\alpha\beta^* - \alpha^*\beta} \text{tr} \left(e^{\beta a^\dagger} e^{-\beta^* a} \rho \right) \quad (1.207)$$

where the average of a normally ordered operator is involved (but contains arbitrarily high moments of the field amplitude). You may find in the literature that the phase factor $e^{\alpha\beta^* - \alpha^*\beta}$ defines a Fourier transformation in the phase space plane.

You may have guessed that the P -function, once you know it, is good for the calculation of expectation values of normally ordered operators $A^{(n)}(a^\dagger, a)$. Using the same reasoning as above, we can write

$$\begin{aligned} A^{(n)}(a^\dagger, a)|\alpha\rangle &= A^{(n)}(a^\dagger, \alpha)|\alpha\rangle \\ \langle\alpha|A^{(n)}(a^\dagger, a)|\alpha\rangle &= A^{(n)}(\alpha^*, \alpha) \\ &= \text{tr} \left(A^{(n)}(\alpha^*, \alpha)|\alpha\rangle\langle\alpha| \right) \\ \langle A^{(n)}(a^\dagger, a) \rangle &= \text{tr} \left(A^{(n)}(a^\dagger, a)\rho \right) = \int d^2\alpha P(\alpha) A^{(n)}(\alpha^*, \alpha) \end{aligned} \quad (1.208)$$

In the end, we get again an average over a c-number function, weighted with $P(\alpha)$ that plays the role of a classical probability distribution (but due to its possibly singularities, it is not one).

Normally ordered operators are important since they correspond to the intensities and correlation functions measured by photodetectors, as you have learned. Whenever the P -function has singularities that do not allow to interpret it as a classical probability distribution, one will get results for photodetector measurements that cannot be explained with “classical statistics” (this amounts to assuming that the field amplitude is randomly distributed). One thus can track down genuine “quantum” features of the states under consideration. In this light, coherent states are really “very classical” as the P -function suggests that their amplitude is precisely fixed.

1.6.3 Examples

In the following, we present some examples where the P -function can be explicitly constructed.

Coherent vs. incoherent sources

We have found earlier that a “classical, monochromatic source”, given by an interaction Hamiltonian $H_{\text{int}} = i\hbar(ga^\dagger - g^*a)$ (written in the rotating frame) creates a coherent state $|\beta\rangle$ when acting for some finite time t , starting from the vacuum state. We found $\beta = gt$, so that the density operator for the final state is $\rho = |\beta\rangle\langle\beta|$.

Now imagine that we do not know the “phase” of the source, i.e., the phase of the complex parameter g . Such a source is called “unphased” or “incoherent”. In this case, all we can do is to use a density matrix where all phase angles of β occur with the same weight $1/2\pi$:

$$\rho = \int \frac{d\varphi}{2\pi} |r_0 e^{i\varphi}\rangle\langle r_0 e^{i\varphi}|$$

The P -function for this state is peaked at the radius $r_0 = |\beta|$ and isotropic

$$P_{r_0}(\alpha) = \frac{1}{2\pi r_0} \delta(|\alpha| - r_0) \quad (1.209)$$

where the radial δ -function is one-dimensional.

Imagine next that we let two sources pump the single mode we have focussed on. If both sources are coherent and create coherent states $|\beta_1\rangle$ and $|\beta_2\rangle$ when acting alone, then letting act the two of them gives:

$$|\psi_{21}\rangle = D(\beta_2)|\beta_1\rangle = D(\beta_2)D(\beta_1)|0\rangle = D(\beta_2 + \beta_1)e^{i\varphi_{12}}|0\rangle$$

The final state is thus the coherent state with the sum amplitude, up to a phase factor (given in (1.203)). This factor drops out when we construct the density matrix, so that we get $\rho_{21} = |\beta_2 + \beta_1\rangle\langle\beta_2 + \beta_1|$. This result is in agreement with our classical intuition: “coherent” sources can be added “coherently” by adding the complex amplitudes. The corresponding photocurrent is

$$\langle I \rangle = \langle a^\dagger a \rangle = \int d^2\alpha |\alpha|^2 P(\alpha) = |\beta_2 + \beta_1|^2$$

since the P -function is δ -peaked at $\beta_2 + \beta_1$.

What do we get when one of the two sources is incoherent? We have to introduce density matrices, and in this section we shall express them in terms of P -functions. It can be shown easily that one gets a density matrix described by

$$P_{21}(\alpha) = \int d^2\alpha' P_2(\alpha - \alpha')P_1(\alpha')$$

when $P_1(\alpha)$ describes the first source and $P_2(\alpha)$ the second. This convolution law is, by the way, identical to what we would obtain from classical statistics: the probability of finding a sum amplitude α is given by the probability of finding amplitudes α' and α'' , provided they have the right sum. Since both sources are assumed to act independently, the probability of finding the two amplitudes factorizes to $P_1(\alpha')P_2(\alpha'')$.

Let us compute the P -function when the first source is coherent, $P_1(\alpha) = \delta^{(2)}(\alpha - \beta_1)$ while the second one is incoherent, as given in (1.209). We get

$$P_{21}(\alpha) = \frac{1}{2\pi r_2} \delta(|\alpha - \beta_1| - r_2)$$

What is the photocurrent in this case?

$$\begin{aligned} \langle I \rangle &= \int d^2\alpha |\alpha|^2 P_{21}(\alpha) \\ &= \int d^2\alpha |\alpha + \beta_1|^2 \frac{1}{2\pi r_2} \delta(|\alpha| - r_2) \end{aligned}$$

In the second line, we changed to the integration variable $\alpha \mapsto \alpha - \beta_1$. Expanding the modulus squared, the mixed term integrates to zero when performing the angular integration (over $\varphi = \arg \alpha$) because it involves $\cos \varphi$ or $\sin \varphi$. The remaining radial integral is trivial because of the δ -function and we find the “incoherently summed intensities”:

$$\langle I \rangle = r_2^2 + |\beta_1|^2 = \langle I_2 \rangle + \langle I_1 \rangle.$$

Chaotic light This light is a generalisation of the thermal blackbody radiation we met earlier. It is a realistic model for most of the light we see outside the laboratory: lightbulbs, neon (gas discharge) lamps etc. All these sources are composed of a large number of elementary sources that radiate

independently of each other. If we feed this light into a single mode, we can compute the P -function using the central limit theorem. Indeed, we saw for two sources that the P -function is like the probability distribution of the amplitude sum of the two sources. For many sources, say N , the sum amplitude α_N is distributed according to a gaussian distribution (this is the central limit theorem) with a mean value given by $\overline{\alpha_N} = \sum_i \overline{\alpha_i}$ and a variance $\Delta\alpha_N^2 = \sum_i \Delta\alpha_i^2$. This gives a P -function

$$P_N(\alpha) = \frac{1}{\pi\Delta\alpha_N^2} e^{-|\alpha - \overline{\alpha_N}|^2 / \Delta\alpha_N^2}.$$

Let us assume for simplicity that all elementary sources are incoherent (then $\overline{\alpha_i} = 0 = \overline{\alpha_N}$). We then get the following mean photon number

$$\langle n \rangle = \int \frac{d^2\alpha}{\pi\Delta\alpha_N^2} |\alpha|^2 e^{-|\alpha|^2 / \Delta\alpha_N^2} = \Delta\alpha_N^2$$

that is given by the variance of the P -function. The P -function for chaotic light is thus of the form

$$P(\alpha) = \frac{1}{\pi\langle n \rangle} e^{-|\alpha|^2 / \langle n \rangle}. \quad (1.210)$$

The corresponding intensity distribution, $p(I) = \langle n \rangle^{-1} e^{-I/\langle n \rangle}$, decays exponentially with a maximum at zero intensity.

It can be shown that in the case $\langle n \rangle$ is the Bose-Einstein mean photon number, we recover the thermal state. The calculation is most easily done by comparing the Q -functions. From the gaussian P -function (1.210), the Q -function is obtained by convolution with another gaussian of width 1. This gives another gaussian with a width increased by one (easy to show taking Fourier transforms). Hence, we find a Q -function

$$Q(\alpha) = \frac{1}{\pi(\langle n \rangle + 1)} e^{-|\alpha|^2 / (\langle n \rangle + 1)}. \quad (1.211)$$

For the thermal density operator (with $\varphi = \hbar\omega/k_B T$), we can directly compute the Q -function and find

$$\begin{aligned} Q_T(\alpha) &= \frac{1}{\pi Z} \langle \alpha | e^{-\varphi a^\dagger a} | \alpha \rangle \\ &= \frac{1}{\pi Z} \sum_{n=0}^{\infty} e^{-\varphi n} |\langle \alpha | n \rangle|^2 \end{aligned}$$

$$\begin{aligned}
&= \frac{e^{-|\alpha|^2}}{\pi Z} \sum_{n=0}^{\infty} \frac{1}{n!} \left(e^{-\varpi} |\alpha|^2 \right)^n \\
&= \frac{\exp[-|\alpha|^2(1 - e^{-\varpi})]}{\pi Z}
\end{aligned}$$

Now the mean thermal photon number satisfies $(\langle n \rangle + 1)^{-1} = 1 - e^{-\varpi}$ and the partition function $Z = (1 - e^{-\varpi})^{-1} = \langle n \rangle + 1$, so that we recover the previous result.

1.6.4 Coherent pumping with damping

As a last example, let us come back to the coherently pumped single-mode field and see how losses relax the mode into a stationary state. We have found before that when the pump continues forever, the mode stays in a coherent state with ever larger amplitude $\alpha = gt$.

Damped field mode

To describe the loss of the field, we need a description that goes beyond the Schrödinger equation. The technical keyword is a ‘master equation’ for the density operator ρ that describes the field mode. We shall see in the next semester how the master equation can be derived; for the moment, we shall just write it down and comment on its implications.

Usually, the equation of motion for the density matrix is given by the commutator with the Hamiltonian H (this is called the ‘von Neumann equation’). We add another term to describe losses that is given by

$$\frac{d\rho}{dt} = -\frac{i}{\hbar} [H, \rho] + \mathcal{L}[\rho] \quad (1.212)$$

$$\mathcal{L}[\rho] = -\frac{\kappa}{2} \{a^\dagger a, \rho\} + \kappa a \rho a^\dagger \quad (1.213)$$

where κ is the damping rate. The object \mathcal{L} is also called a Liouvillian or a ‘superoperator’.

We use the Hamiltonian

$$H = -\hbar \Delta a^\dagger a + i\hbar (g a^\dagger - g^* a)$$

where $\Delta = \omega_p - \omega_m$ is the detuning between the pump and mode frequencies.

Let us first explore the consequences of the Liouvillian.

Amplitude damping

The field amplitude corresponds to the operator a . Its expectation value decays exponentially (the subscript ‘damp’ means: we only use the time-dependence due to the damping Liouvillian)

$$\begin{aligned} \left. \frac{d}{dt} \right|_{\text{damp}} \langle a \rangle &= \text{tr} (a \mathcal{L} \rho) \\ &= -\frac{\kappa}{2} \text{tr} (aa^\dagger a \rho + a^\dagger a a \rho - 2a^\dagger a a \rho) \\ &= -\frac{\kappa}{2} \text{tr} (a \rho) = -\frac{\kappa}{2} \langle a \rangle \end{aligned} \quad (1.214)$$

In the second step, we have cyclically permuted the operators under the trace.⁶ The next step uses the commutation relation between a and a^\dagger .

A similar calculation gives for the mean photon number

$$\left. \frac{d}{dt} \right|_{\text{damp}} \langle a^\dagger a \rangle = -\kappa \langle a^\dagger a \rangle. \quad (1.215)$$

It is reasonable that the damping master equation yields these two properties; but they alone do not completely specify the Liouvillian superoperator. (Exercise: exchange a and a^\dagger and the sign of κ .) A much stronger constraint comes from the requirement that a density matrix ρ retains its basic properties. This implies that the trace of ρ is unchanged: it is equal to the probability (=1) of finding the quantum system in *any* state. It can be easily checked, using the same manipulations as above that indeed, $\text{tr} \mathcal{L} \rho = 0$. Another requirement is that the eigenvalues of ρ remain positive.

Quantum Monte Carlo ‘unravelling’

Finally, we illustrate a method to interpret the master equation in terms of measurements on selected observables of the system. We stick in our description to the specific example of a single field mode in a cavity. The quantity that is monitored are the photons that leak to the outside world.

Consider a time interval dt , and assume that there is a probability γdt that a photon leaks out of the cavity. Then, we know that the cavity mode

⁶This is possible if the trace exists for the operator product or in other words, ρ is a ‘trace-class’ operator.

contains one photon less. In terms of a trace-normalized density matrix, the ‘projection postulate’ (*Kollaps der Wellenfunktion*) leads to

$$\text{prob } \gamma dt : \quad \rho \mapsto \frac{a\rho a^\dagger}{\text{tr}(a^\dagger a \rho)} \quad (1.216)$$

This process is also called a ‘quantum jump’, and a is the ‘jump operator’ (annihilating one photon in the system). But also when no photon is observed during dt , we gain information about the system: qualitatively speaking, our tendency of assuming that there are no or very few photons in the cavity will increase. This information gain can also be translated into a change of the density operator: those basis states with a large photon number will have a decreased probability. This change can be achieved with the (trace normalized) mapping

$$\text{prob } 1 - \gamma dt : \quad \rho \mapsto \frac{(1 - \eta a^\dagger a)\rho(1 - \eta a^\dagger a)}{\text{tr}((1 - \eta a^\dagger a)^2 \rho)} \quad (1.217)$$

where η is a small parameter to be specified. Now, we join together these two alternative outcomes of the measurement into an ensemble described by a classical mixture of density matrices (in the same way as mixed quantum states are described by a density matrix):

$$\begin{aligned} \rho(t + dt) = & \gamma dt \frac{a\rho a^\dagger}{\text{tr}(a^\dagger a \rho)} \\ & + (1 - \gamma dt) \frac{(1 - \eta a^\dagger a)\rho(1 - \eta a^\dagger a)}{\text{tr}((1 - \eta a^\dagger a)^2 \rho)} \end{aligned} \quad (1.218)$$

Now, we have to specify the probability γdt : it is proportional to the rate κ at which photons leave the cavity and to the photon number $\langle a^\dagger a \rangle = \text{tr}(a^\dagger a \rho)$: ‘each photon decides on its own whether it will leave the cavity or not’.⁷ Hence, $\gamma = \kappa \text{tr}(a^\dagger a \rho)$, and the first term in Eq.(1.218) reduces to the ‘sandwich term’ $\kappa a \rho a^\dagger$ of the Louvillian.

A similar cancellation

$$1 - \gamma dt = 1 - \kappa \langle a^\dagger a \rangle dt = \text{tr}((1 - \eta a^\dagger a)^2 \rho) \quad (1.219)$$

⁷The same reasoning also leads in radioactive decay to the exponential decay of the number of excited nuclei.

occurs, to first order in dt and η , with the choice $\eta = \frac{1}{2}\kappa dt$. To first order in dt and in η , we thus get the differential equation:

$$\frac{\rho(t + dt) - \rho(t)}{dt} = \kappa a \rho a^\dagger - \frac{\kappa}{2} \{a^\dagger a \rho + \rho a^\dagger a\} \quad (1.220)$$

which is the damping master equation.

The key point of this interpretation is that after the time step dt , the density matrix is a classical mixture (a ‘convex sum’ with positive coefficients) of two terms that are both density matrices. Strictly speaking, this does not apply for arbitrarily high Fock states $|n\rangle$ since $1 - \kappa\langle\hat{n}\rangle dt$ can become negative. More precise derivations use, e.g., a positive super-operator in the form of an exponential, $\exp(-\frac{1}{2}\kappa\hat{n})\rho\exp(-\frac{1}{2}\kappa\hat{n})$.

Master equation for P -function

From the master equation, we now want to find the equation of motion for the P -function. To this end, we use its definition: the density matrix is expanded in projection operators $|\alpha\rangle\langle\alpha|$. Let us work out the action of the different operators in the master equation on these projectors.

$$[a^\dagger a, |\alpha\rangle\langle\alpha|] = \alpha a^\dagger |\alpha\rangle\langle\alpha| - \alpha^* |\alpha\rangle\langle\alpha| a$$

This was easy. Now we have to get an idea how the creation operator acts on a coherent state. Using the displacement operator, we find

$$a^\dagger D(\alpha) = D(\alpha) a^\dagger - \alpha^* [a^\dagger, a] D(\alpha) = D(\alpha) a^\dagger + \alpha^* D(\alpha)$$

On the other hand, we can write

$$D(\alpha) a^\dagger |0\rangle = \frac{\partial}{\partial \alpha} D(\alpha) |0\rangle$$

where the derivative is to be taken as if α and α^* were independent variables (and α^* is held constant). With the same convention, we can affirm

$$\frac{\partial}{\partial \alpha} \langle\alpha| = 0.$$

(In the derivative of the displacement operator, only a^\dagger occurs, and gives zero when acting to the left on the vacuum state.)

We finally get

$$a^\dagger |\alpha\rangle\langle\alpha| = \left(\alpha^* + \frac{\partial}{\partial\alpha} \right) |\alpha\rangle\langle\alpha|. \quad (1.221)$$

In a similar way, one can show that

$$|\alpha\rangle\langle\alpha| a = \left(\alpha + \frac{\partial}{\partial\alpha^*} \right) |\alpha\rangle\langle\alpha|, \quad (1.222)$$

and the required commutator simplifies to

$$[a^\dagger a, |\alpha\rangle\langle\alpha|] = \left(\alpha \frac{\partial}{\partial\alpha} - \alpha^* \frac{\partial}{\partial\alpha^*} \right) |\alpha\rangle\langle\alpha|.$$

The other terms in the master equation can be handled in the same way. We thus find

$$\begin{aligned} \int d^2\dot{P}(\alpha) |\alpha\rangle\langle\alpha| &= \int d^2P(\alpha) \left(i\Delta \left(\alpha \frac{\partial}{\partial\alpha} - \alpha^* \frac{\partial}{\partial\alpha^*} \right) + \left(g \frac{\partial}{\partial\alpha} + g^* \frac{\partial}{\partial\alpha^*} \right) \right. \\ &\quad \left. - \frac{\kappa}{2} \left(\alpha \frac{\partial}{\partial\alpha} + \alpha^* \frac{\partial}{\partial\alpha^*} \right) \right) |\alpha\rangle\langle\alpha| \end{aligned}$$

Now we make a partial integration, assuming that $P(\alpha)$ drops sufficiently fast at infinity that we can drop the boundary terms. This makes the derivatives act on $P(\alpha)$. Finally, we compare the coefficients in the expansion in coherent states.⁸ We thus find the equation

$$\dot{P}(\alpha) = \frac{\partial}{\partial\alpha} \left[\left(-i\Delta\alpha - g + \frac{\kappa}{2}\alpha \right) P(\alpha) \right] + \frac{\partial}{\partial\alpha^*} \left[\left(i\Delta\alpha - g^* + \frac{\kappa}{2}\alpha^* \right) P(\alpha) \right]$$

which is equivalent to the master equation.

Stationary state

Let us look at its solution in steady state. If our intuition can be trusted, we expect the stationary mode amplitude to be finite. The equation can be fulfilled with the following *Ansatz*:

$$\begin{aligned} \left(-i\Delta\alpha - g + \frac{\kappa}{2}\alpha \right) P(\alpha) &= \text{const.}(\alpha^*) \\ \left(i\Delta\alpha - g^* + \frac{\kappa}{2}\alpha^* \right) P(\alpha) &= \text{const.}(\alpha) \end{aligned} \quad (1.223)$$

⁸This uses the fact — not demonstrated here, but apparent from (1.207) — that the expansion of a state in P -functions is unique.

where the constants may depend on α^* or α . Note that the second condition is the complex conjugate of the first. A first guess of solution is given by choosing a nonzero constant, leading to

$$\begin{aligned} P^{(st)}(\alpha) &= \frac{N}{-i\Delta\alpha - g + \frac{\kappa}{2}\alpha} \frac{1}{i\Delta\alpha^* - g^* + \frac{\kappa}{2}\alpha^*} \\ &= \frac{N}{\Delta^2 + \kappa^2/4} \frac{1}{|\alpha - \alpha_s|^2} \quad (\text{wrong}) \end{aligned}$$

which is peaked at the stationary amplitude given by

$$\alpha_s = \frac{g}{\kappa/2 - i\Delta} \quad (1.224)$$

But this solution is ill: it is not normalizable because the singularity at $\alpha = \alpha_s$ is too strong.

Our only choice is to choose a constant which is zero in (1.223). This leads us to the condition

$$(\alpha - \alpha_s)P(\alpha) = 0$$

that also fulfills the complex conjugate condition. The solution of this equation is ... a $\delta^{(2)}$ -function. With the correct normalization, we have

$$P^{(st)}(\alpha) = \delta^{(2)}(\alpha - \alpha_s). \quad (1.225)$$

The field mode thus ends up in a coherent state with an amplitude $\alpha_s = g/(\kappa/2 - i\Delta)$ given by the ratio between the pump rate g and the loss rate κ (on resonance). This is typical for a steady state solution where “gain” is compensated by “loss”, and we shall encounter other examples in laser theory.

We also note that the “zero constant” option we were forced to take also occurs generically. One can show that this is the only possibility to obtain a normalizable solution.