Quanten-Informatik und Theoretische Quantenoptik I

Wintersemester 2009/10
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Übungsaufgaben Blatt 1 Ausgabe: 19. Oktober 2009
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Hinweis. Lassen Sie sich von Fehlern in den angegebenen Formeln nicht verwirren. Im Zweifelsfall ("das Ergebnis sieht aber kompliziert aus") fehlt eben im Aufgabentext ein Faktor 2, π, i, -1 ...

Problem 1.1 – Information and thermodynamics (5 points)

Consider a quantum system with discrete states described by the density matrix ρ (a hermitean, positive definite matrix with trace $\mathrm{tr}\,\rho = 1$). The *von Neumann entropy* of ρ is defined by ($k_B = 1$)

$$S_{\mathrm{vN}}(\rho) = -\mathrm{tr}\,(\rho \log \rho) \tag{1.1}$$

where $\log \rho$ is the matrix logarithm. (i) Show that $S_{\mathrm{vN}}(\rho) \leq \log 2$ for a two-state system; the maximum is reached if both states occur with equal probability.

(ii) The *thermodynamical entropy* is given in terms of the canonical partition function by

$$S_{\mathrm{tdyn}} = \frac{\partial}{\partial T}\,(T \log Z) \tag{1.2}$$

where Z is the partition function (*kanonische Zustandssumme*). Show that this entropy coincides with the von Neumann entropy for a system in thermal equilibrium.

Note. The partition function in the canonical ensemble is defined by $Z = \mathrm{tr}\,\exp(-H/T)$ where T is the temperature and H the system Hamiltonian (a hermitean, positive semi-definite matrix). In thermal equilibrium, the system density matrix is $\rho = Z^{-1}\,\exp(-H/T)$.

Problem 1.2 – Typical numbers (4 points)

(i) Calculate the z-component of the electric dipole matrix element d^z_{eg} in the hydrogen atom between the states $|\mathrm{e}\rangle = |2\mathrm{p}_z\rangle$, $|\mathrm{g}\rangle = |1\mathrm{s}\rangle$. Look up the hydrogen wave functions in your favorite quantum mechanics text. The order of magnitude is ea_0 [a_0: Bohr radius], of course.

Note. The operator for the electric dipole moment in the hydrogen atom is $d^z = ez$ where z is the electron coordinate relative to the proton. The matrix element is $d^z_{\mathrm{eg}} = \langle \mathrm{e}|d_z|\mathrm{g}\rangle$.

(ii) In the lecture, we have seen the Rabi frequency Ω for the electric-dipole coupling of a two-level atom to an electric field,

$$\frac{\hbar\Omega}{2} = -\mathbf{d}_{\mathrm{eg}} \cdot \mathbf{E}(\mathbf{r}_A) \tag{1.3}$$

Estimate its order of magnitude for the electric field $\mathbf{E}(\mathbf{r})$ of a hand-held laser pointer. (Ask around or think yourself about the typical parameters: light intensity, beam diameter etc.) Compare the interaction energy $\hbar\Omega$ to typical energy differences between stationary energy levels of hydrogen.

Problem 1.3 – Minimal coupling vs dipole coupling (6 points)

(i) Write down the minimal coupling Hamiltonian for an electron in an electromagnetic field (scalar and vector potential). Expand to lowest order in the vector potential and show that in the Coulomb gauge (i.e., $\nabla \cdot \mathbf{A} = 0$), you get as interaction Hamiltonian the so-called "p.A coupling":

$$H_{\mathrm{pA}} = -\frac{e}{m}\mathbf{p} \cdot \mathbf{A}(\mathbf{x}) \tag{1.4}$$

where $\mathbf{x}$ and $\mathbf{p}$ are the position and momentum operators of the electron. Why is the operator ordering irrelevant here?

(ii) Specialize to an electron in a given electromagnetic field [$\mathbf{A}(\mathbf{x}) \mapsto \mathbf{A}(\mathbf{x}, t)$, a classical time-dependent vector field, evaluated at the electron position operator] and make the long-wavelength approximation $\mathbf{A}(\mathbf{x}, t) \approx \mathbf{A}(\mathbf{0}, t)$ [$\mathbf{0}$: origin, 'center of the atom']. Perform the following unitary transformation on the electron wave function

$$\psi(\mathbf{x}, t) \mapsto \tilde{\psi}(\mathbf{x}, t) = \exp\left[-\frac{\mathrm{i}}{\hbar} e\chi(\mathbf{x}, t)\right] \psi(\mathbf{x}, t), \tag{1.5}$$

with the "gauge function"

$$\chi(\mathbf{r}, t) = (\mathbf{r} - \mathbf{0}) \cdot \mathbf{A}(\mathbf{0}, t) \tag{1.6}$$

Calculate the Schrödinger equation for the 'new wave function' $\tilde{\psi}(\mathbf{x}, t)$ and check that the Hamilton operator in this equation does not depend on the vector potential $\mathbf{A}(\mathbf{0}, t)$, but on its time derivative $\mathbf{E} = -\partial_t \mathbf{A}$. The interaction Hamiltonian now contains the electric dipole coupling $-e(\mathbf{r} - \mathbf{0}) \cdot \mathbf{E}(\mathbf{0}, t)$.

(iii) [2 bonus points] Comment on the name 'gauge function'.

Problem 1.4 – Projekt Wikipedia (5 points)

[Abgabe per email bis zum 29. Oktober 2009]

(i) Schlagen Sie den Eintrag "Quantenoptik" auf und vergleichen Sie inhaltlich mit den Einträgen in anderen Sprachen. Diskutieren Sie mit Ihren Kommilitonen und erstellen Sie eine Liste von Überschriften: welche Information würden Sie unter dem Titel "Quantenoptik" lesen wollen?

(ii) Sichten Sie die Diskussionsbeiträge des letzten Potsdamer Kurses zur Quantenoptik (Wikipedia Diskussion und Potsdamer Web-Site vom SS 09). Erstellen Sie eine Liste von weiteren Unterpunkten, die Sie für interessant halten.