Quanten-Informatik und Theoretische Quantenoptik I

Wintersemester 2009/10

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Übungsaufgaben Blatt 2

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Problem 2.1 – Rabi oscillations and dipole traps (6 points)

So far you have seen Rabi oscillations on resonance ($\Delta = \omega_L - \omega_A = 0$), but using red- or blue-shifted light is very useful. See why!

(i) Solve the two-level dynamics $c_e(t), c_g(t)$ with nonzero detuning $\Delta = \omega_L - \omega_A$ in the rotating-wave approximation (RWA). The Hamiltonian is

$$H_{\text{RWA}} = \frac{\hbar\omega_A}{2}\sigma_3 + \frac{\hbar\Omega}{2}\left(\sigma^\dagger\,\mathrm{e}^{-\mathrm{i}\omega_L t} + \sigma\,\mathrm{e}^{\mathrm{i}\omega_L t}\right) \tag{2.1}$$

where the Rabi frequency Ω is chosen real. You might find it useful to introduce the effective Rabi frequency $\Omega_R = \sqrt{\Delta^2 + \Omega^2}$. Work out the solution for an atom initially prepared in the ground state and in the excited state.

(ii) Interpret the average atomic energy (time-averaged over a Rabi period) of an atom as an effective potential

$$V_{\text{eff}} = -\frac{\hbar\Delta}{2}(\langle|c_e(t)|^2\rangle_t - \langle|c_g(t)^2|\rangle_t)\ . \tag{2.2}$$

(Why is the energy scale $\hbar\Delta$ rather than $\hbar\omega_A$?) Discuss the forces acting on atoms initially in the ground and excited state placed in an inhomogeneous electric field ('optical tweezer', 'optical lattice'). How does the sign of Δ matter?

Problem 2.2 – Typical numbers (4 points)

(i) In the last problem set you have estimated the electric field of a typical laser pointer. How strong is its magnetic field? Estimate the electric and the magnetic interaction energies of an atom in a typical laser field.

(ii) Derive from the time-dependent amplitudes given in the lecture the frequency-dependent polarizability $\alpha(\omega)$ of a two-level atom. You need the expectation value of the dipole operator

$$\mathbf{d} = \mathbf{d}_{\text{eg}}|\text{e}\rangle\langle\text{g}| + \mathbf{d}_{\text{ge}}|\text{g}\rangle\langle\text{e}| \tag{2.3}$$

more precisely its Fourier component proportional to $\mathrm{e}^{-\mathrm{i}\omega_L t}$:

$$\langle\mathbf{d}\rangle = \alpha(\omega_L)\mathbf{E}\,\mathrm{e}^{-\mathrm{i}\omega_L t} + \text{c.c.} \tag{2.4}$$

This equation defines the (linear) polarizability (usuall a tensor), with $\mathbf{E}$ the complex amplitude of the field. The result for an atom initially in the ground state is

$$\alpha(\omega) = \frac{(2\omega_A/\hbar)\mathbf{d}_{\text{ge}}\mathbf{d}_{\text{eg}}}{\omega_A^2 - \omega^2} \tag{2.5}$$

(iii) Estimate the electrostatic polarization energy. (**Hint.** $\alpha(0)/\varepsilon_0 \sim a_{\text{Bohr}}^3$.)

(iv) [2 bonus points] Speculate about the electric field in a laser beam and its relation to the photon number.

Problem 2.3 – Non-resonant Hamiltonian (5 points)

In the lecture, we argued that non-resonant part of the Hamiltonian can be neglected (the rotating-wave approximation). See here how accurate this approximation is.

(i) Calculate the two-level dynamics, but for the non-resonant coupling only: re-do the time-dependent perturbation theory outlined in the lecture and give an upper bound to $|c_e(t)|^2$ for an atom initially in the ground state.

(ii) Estimate the ground state energy shift and compare your result to problem 2.1.

Problem 2.4 – Quantum states of a qubit (5 points)

The density operator for a two-level system is a hermitean 2×2 matrix with trace one.

(i) Consider a density matrix of the form

$$\rho = \frac{1}{2}\left(\mathbb{1} + \sum_{i=1}^{3} \sigma_i s_i\right). \tag{2.6}$$

Show that the vector with components s_i (called the *Bloch vector*) is equal to the expectation value of the operators σ_i and that its components are real. Recall that the Pauli matrices σ_i ($i = 1, 2, 3$) satisfy the relation $\mathrm{tr}(\sigma_i\sigma_j) = 2\delta_{ij}$.

(ii) A quantum state described by the density matrix ρ is called *pure* if $\mathrm{tr}\,\rho^2 = 1$. This is true for any dimension of the system. Conclude that the density matrix of a pure state can be written as $\rho = |\psi\rangle\langle\psi|$ (i.e., in diagonal form with a single eigenvalue one).

(iii) Show that the two-level system is in a pure state if and only if the Bloch vector is on the unit sphere: $\sum_i s_i^2 = 1$.