Quanten-Informatik und Theoretische Quantenoptik I

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Problem 3.1 – Transverse delta function (6 points)
In the lecture, we have encountered the so-called 'transverse δ-function' $\delta^{\perp}_{ij}(\mathbf{x})$. It has the following properties ($\mathbf{F}(\mathbf{x})$ is an arbitrary vector field)

$$\text{if } \mathbf{F}(\mathbf{x}) \text{ is transverse,} \qquad \int \mathrm{d}^3x' \sum_j \delta^{\perp}_{ij}(\mathbf{x}-\mathbf{x}')F_j(\mathbf{x}') = F_i(\mathbf{x}) \tag{3.1}$$

$$\text{if } \mathbf{F}(\mathbf{x}) = \nabla\phi(\mathbf{x}), \qquad \int \mathrm{d}^3x' \sum_j \delta^{\perp}_{ij}(\mathbf{x}-\mathbf{x}')F_j(\mathbf{x}') = 0 \tag{3.2}$$

$$\delta^{\perp}_{ij}(\mathbf{x}) = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} (\delta_{ij} - \frac{k_i k_j}{k^2}) \tag{3.3}$$

$$\delta^{\perp}_{ij}(\mathbf{x}) = \delta_{ij}\delta(\mathbf{x}) + \frac{1}{4\pi}\frac{\partial^2}{\partial x_i \partial x_j}\frac{1}{|\mathbf{x}|} \tag{3.4}$$

(1) Show that Eq.(3.3) correctly implements the properties (3.1, 3.2) by working with the spatial Fourier transform of the vector function $\mathbf{F}(\mathbf{x})$.

(2) Prove Eq.(3.4) by analogy to a gauge transformation: Be $\mathbf{F}(\mathbf{x})$ any (sufficiently smooth) vector field. Find a 'gauge function' $\chi(\mathbf{x})$ such that $\mathbf{F}'(\mathbf{x}) := \mathbf{F}(\mathbf{x}) + \nabla\chi(\mathbf{x})$ is transverse.

Hint. The 'gauge function' is the solution to an inhomogeneous Laplace equation.

Problem 3.2 – Commutator of fields (7 points)
In the lecture, we have found that the commutator between the quantized vector potential $\mathbf{A}$ and its conjugate momentum field $\mathbf{\Pi}$ should be

$$[A_i(\mathbf{x}), \Pi_j(\mathbf{x}')] = \mathrm{i}\hbar\delta^{\perp}_{ij}(\mathbf{x}-\mathbf{x}') \tag{3.5}$$

(i) Take the rotation with respect to x and the index i and observe that this is nonzero. Show that one gets the so-called Pauli commutator between the (transverse) fields $\mathbf{E}$ and $\mathbf{B}$:

$$[B_i(\mathbf{x}), E_j(\mathbf{x}')] = \frac{\mathrm{i}\hbar}{\varepsilon_0}\epsilon_{ijk}\frac{\partial}{\partial x_k}\delta(\mathbf{x}-\mathbf{x}') \tag{3.6}$$

(ii) In a mathematically careful (quantum) field theory, the fields $\mathbf{A}$, $\mathbf{E}$, $\mathbf{B}$ etc are actually (operator-valued) distributions that have to be 'smeared out' with suitable smooth test functions $\mathbf{f}(\mathbf{x})$, $\mathbf{g}(\mathbf{x})$. This gives operators that behave in a less singular way; in particular you can multiply fields evaluated at the same point. Consider thus the observables

$$\mathcal{E}[\mathbf{f}] = \int \mathrm{d}^3x\, \mathbf{f}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}), \qquad \mathcal{B}[\mathbf{g}] = \int \mathrm{d}^3x\, \mathbf{g}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x}) \tag{3.7}$$

Work out the commutator $[\mathcal{E}[\mathbf{f}], \mathcal{B}[\mathbf{g}]]$ and conclude that orthogonal components of $\mathbf{E}$ and $\mathbf{B}$ at neighboring points cannot be measured simultaneously. Derive, as in the quantum mechanics I course, the uncertainty relation between the variances $(\Delta\mathcal{E}[\mathbf{f}])^2$, $(\Delta\mathcal{B}[\mathbf{g}])^2$.

(iii) [4 bonus points] In the construction of (iii), focus on mode functions of the form $\nabla \times \mathbf{g} = (\omega/c)\mathbf{f}$ with $\mathbf{f}$ being normalized as $\int \mathrm{d}^3x\, \mathbf{f}^2(\mathbf{x}) = 1$ and $\omega > 0$ a constant. Consider a quantum state for the field where the average values of $\mathcal{E}[\mathbf{f}]$, $\mathcal{B}[\mathbf{g}]$ are zero and their variances are identical. What do you get then for the 'energy' $\varepsilon_0\langle\mathcal{E}[\mathbf{f}]^2\rangle + (1/\mu_0)\langle\mathcal{B}[\mathbf{g}]^2\rangle$?

Answer: One 'photon energy' $\hbar\omega$ spread over the spatial size of the test function $\mathbf{f}(\mathbf{x})$.

Problem 3.3 – Quantum logic (7 points)

Quantum computer scientists are very fond of two-level states, which they use as a qbit and sometimes represent on the Bloch sphere. Quantum computing[1] is then done by applying all kind of useful unitary operators.

(i) A general class of unitary operators is given by

$$R(\vec{n}, \theta) = \exp(i\theta\vec{n} \cdot \vec{\sigma}/2). \tag{3.8}$$

where $\vec{n}$ is a unit vector and θ an angle. Check explicitly that this operator rotate the the Bloch vector associated to a Hilbert space element. Confirm that opposite points on the Bloch sphere are orthogonal in Hilbert space.

(ii) Meet another favorite unitary operator acting on two-level states, the Hadamard gate. It is given by

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} . \tag{3.9}$$

What is its action in Hilbert space and on the Bloch sphere? Can you express it in the form $R(\vec{n}, \theta)$ of Eq.(3.8)? Why is the Hadamard gate called $\sqrt{\mathbb{1}}$-gate?

(iii) [4 Bonus points] Quantum calculations must not destroy information, which would produce heat and possibly destroy the quantum computer[2]. The AND-gate requires the input of two quantum states and gives only one as output. Can you estimate how much entropy is produced?

[1] R. Feynman, *Quantum Mechanical Computers*, Found. Phys. **16** (1986).

[2] Bennett, IBM J. Res. Dev. **17** (1973), available at www.dha.caltech.edu

(iv) A classical logical operator that can be implemented in quantum computing is the NOT gate acting on one qbit. Explain, why it is given by the Pauli matrix σ_1. What is its action in Hilbert space and on the Bloch sphere? Quantum computer scientists love to represent it as a concatenation of two $\sqrt{NOT}$ gates. Use (i) to find a representation.