Quanten-Informatik und Theoretische Quantenoptik I

Wintersemester 2009/10
Carsten Henkel/Axel Pelster

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Problem 4.1 – Photon spin (5 points)

Make yourself acquainted with the field theory version of Emmy Noether's theorem[3]. Please understand how the spin angular momentum of the Maxwell field is given by

$$\vec{S}(t) = \int d^3\vec{x}\, \vec{A}_\perp(\vec{x}, t) \times \vec{\Pi}(\vec{x}, t). \tag{4.1}$$

i) By correspondence, the spin angular momentum operator reads

$$\hat{\vec{S}}(t) = \frac{1}{2} \int d^3\vec{x}\, \left[\hat{\vec{A}}_\perp(\vec{x}, t) \times \hat{\vec{\Pi}}(\vec{x}, t) - \hat{\vec{\Pi}}(\vec{x}, t) \times \hat{\vec{A}}_\perp(\vec{x}, t) \right] \tag{4.2}$$

Is the operator ordering really relevant here? Use the canonical field quantization to find a mode expansion of $\hat{\vec{S}}(t)$.

ii) Now, what is the spin of the photon field? Recall what you know about the photon field Lagrangian to draw conclusions on its helicity.

Problem 4.2 – Fock space, bosons, fermions, and temperature (6 points)

In the lecture we have quantized the photon field which adheres to bosonic commutation relations. Let us meditate on the structure of the Fock space for both fermions and bosons that arises immediately from the action of creation and annihilation operators.

(i) Let $\{n\} = \{n_1, \ldots n_k, \ldots\}$ stand for the set of occupation numbers in all (photon) field modes. Show that for a stationary bosonic state, one has

$$\langle \{n\} | a_k^\dagger a_l | \{n\} \rangle = \delta_{kl} n_k, \qquad \langle \{n\} | a_k a_l^\dagger | \{n\} \rangle = \delta_{kl}(n_k + 1). \tag{4.3}$$

where n_k is the photon occuation number in mode k. Show furthermore, that all other products of one or two mode operators have zero expectation value.

(ii) Find the canonical partition function (Boltzmann-weighted average)

$$Z_N^\pm = \mathrm{Tr}_{\mathcal{H}_N} \exp(-\beta H) \tag{4.4}$$

[3] e.g., W. Greiner, J. Reinhardt, *Theoretische Physik Band 7a – Feldquantisierung* or Dr. Blümlein's Quantum Field Theory lecture.

for bosons (+) / fermions (−) where the trace $\mathrm{Tr}_{\mathcal{H}_N}$ is performed in the N-particle Hilbert space $\mathcal{H}_N$. Use this result to calculate the grandcanonical partion function

$$\mathcal{Z}^{\pm} = \mathrm{Tr}_{\mathcal{F}} \exp(-\beta[H - \mu N]) \tag{4.5}$$

Think about how the trace is performed in Fock-space $\mathcal{F}$ (number eigenstates)! μ is of course the chemical potential and N the photon number operator.

(iii) The average occupation number is then given by $\langle n^{\pm} \rangle = \partial_\mu \beta^{-1} \ln \mathcal{Z}^{\pm}$. This must of course recover the Fermi-Dirac / Bose-Einstein statistics.

(iv) Back to photons: Why is there no photon chemical potential (Planck statistics)? Use your results to express the thermal commutators for the photon field

$$\langle a_k^\dagger a_l \rangle_T, \qquad \langle a_k a_l^\dagger \rangle_T. \tag{4.6}$$

Problem 4.3 – What is a photon? (5 points)

The choice of an orthonormal basis (ONB) of mode functions is a mere matter of taste. Plane waves are certainly convenient but of course they are not the only possible choice.

(i) Find a representation of the field operator in terms of spherical harmonics. Please express a plane wave one-photon state $a_k^\dagger|0\rangle$ in this new basis. (If you prefer, you can ignore the photon polarization for a moment. For 3 bonus points, work with the 'vector spherical harmonics'.)

(ii) Comment on the operator 'photon number per mode', $n_k = a_k^\dagger a_k$, and the total number operator $N = \sum_k n_k$. Consider a change of the ONB. What does the choice of the ONB mean for the photon concept? What is a photon?

Problem 4.4 – Vacuum energy (4 points)

Normal ordering gets rid of a potentially divergent vacuum energy contribution (i.e., the sum of $\hbar\omega_k$ over all modes k). This energy has important implications, among which are the Casimir effect and possibly dark energy in the universe.

To estimate the vacuum energy of the universe, find a cut off so that the vacuum energy density equals the critical density of the universe $\approx 10^{-29}\,\mathrm{g\,cm^{-3}}$. An intuitive cut off is given by wavelengths corresponding to the Planck scale $10^{-35}\,\mathrm{m}$, where the geometric structure of spacetime itself becomes a quantum field. Compare the two results.