Quanten-Informatik und Theoretische Quantenoptik I

Wintersemester 2009/10
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Problem 7.1 – Spontaneous emission (10 points)
In the lecture, we have found the following expression for the probability $p_e(t)$ to find a two-level atom in its excited state

$$p_e(t) \approx 1 - \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{\omega_k}{2\hbar\varepsilon_0} \sum_\lambda |\mathbf{e}^*_{\mathbf{k}\lambda} \cdot \mathbf{d}|^2 \frac{4\sin^2[(\omega_k - \omega_A)t/2]}{(\omega_k - \omega_A)^2} \tag{7.1}$$

where $\mathbf{d}$ is the transition dipole, $\mathbf{e}_{\mathbf{k}\lambda}$ the unit polarization vector for the plane-wave mode with polarization λ, wave vector $\mathbf{k}$ and frequency ω_k.

(a) Compute the integral over the angles θ, φ of $\mathbf{k}$ and the sum over λ to show that

$$\int \frac{\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\varphi}{4\pi} \sum_\lambda |\mathbf{e}^*_{\mathbf{k}\lambda} \cdot \mathbf{d}|^2 = \frac{2}{3}|\mathbf{d}|^2 \tag{7.2}$$

If you wish, you may assume that $\mathbf{d}$ is a real vector.

(b) For $t \to 0$, the probability $p_e(t)$ is $1 + \mathcal{O}(t^2)$ to leading order. Estimate the prefactor for this quadratic law: assume a high-frequency cutoff $\omega_c \sim c/a_0$. For which times t does the quadratic correction become comparable to 1?

(c) Insert a gaussian cutoff function $\exp[-\frac{1}{2}(\omega_k/\omega_c)^2]$ into the k-integral remaining in Eq.(7.1) and evaluate the integral exactly. Identify a 'window' of time scales where a linear dependence takes over so that the decay rate $\gamma_e = -\mathrm{d}p_e/\mathrm{d}t$ can be defined in a meaningful way. You may use the approximation $\omega_A \ll \omega_c$ to simplify the calculation.

(d) [5 bonus points] The limit of long times: in some models of spontaneous decay, the probability $p_e(t)$ shows an algebraic tail (i.e., an inverse power law $t^{-\nu}$ with some exponent $\nu > 0$). This is due to the fact that as a function of ω_k, the integral is restricted to $\omega_k \geq 0$, the integrand showing a discontinuity in some higher derivative at $\omega_k = 0$. Can you reproduce this behaviour from the integral (7.1), possibly combined with a gaussian cutoff?

Problem 7.2 – A virtual photon cloud around an atom (5 points)
As a variation to the calculation of spontaneous emission, consider the states $|g, \mathrm{vac}\rangle$ and $|e, 1_{\mathbf{k}\lambda}\rangle$ and a coupling between atom and field without making the resonance (or rotating-wave) approximation:

$$H_{\mathrm{int}} = -\sum_{\mathbf{k}\lambda} E_k \left(\mathbf{e}_{\mathbf{k}\lambda}\, a_{\mathbf{k}\lambda} + \mathbf{e}^*_{\mathbf{k}\lambda}\, a^\dagger_{\mathbf{k}\lambda}\right) \cdot \left(\mathbf{d}\sigma + \mathbf{d}^*\sigma^\dagger\right) \tag{7.3}$$

where E_k is the electric field per photon. (a) Compute the matrix elements of H_{int} between these states.

(b) Calculate in stationary (i.e., conventional) first-order perturbation theory the eigenstate $|\psi^{(1)}\rangle$ that connects for $H_{\text{int}} \to 0$ to the ground state $|g, \text{vac}\rangle$ and compute its average photon number. Can you give a 'spectral expansion' of this photon number? What about the convergence at low and high frequencies?

Problem 7.3 – Quantum states of single-mode fields (5 points)

In the lecture, we have encountered Fock (or number) states $|n\rangle$ and Boltzmann (or thermal) states $\hat{\varrho}_T$ of a single radiation mode. The Glauber (or coherent) states $|\alpha\rangle$ (α is a complex number) are defined as

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{7.4}$$

Compute the average photon number $\langle \hat{n} \rangle$ and its standard deviation Δn for these three states as a function of the parameters n, T (temperature) and α, respectively.