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The preliminary programme for this lecture:

1. Atom-field dynamics

- atom-light coupling
- the model of a two-level atom
- quantum states of one and two qubits, entanglement
- Bloch equations and Rabi oscillations

2. Field quantization: QED

- canonical scheme in the Coulomb gauge
- quantum (or vacuum) fluctuations
- states of the radiation field: Fock, coherent, thermal, squeezed; distribution functions in phase space
- state transformation at a beamsplitter, optical quantum computing

3. Applications

- cavity QED: Jaynes-Cummings model, collapse and revival of Rabi oscillations
- quantum gates with atoms and cavity photons
- master equations for spontaneous decay and photodetection

# Chapter 1

## Atom-light interaction

In this chapter, we review some basic physics about atoms and their interaction with light. The focus is on developing simple approximations that describe the coupling to near-resonant light fields.

To begin with, we describe in this chapter the field classically. The argument can be done as well, however, with the quantized field. We shall give here and there the corresponding formulas without going into the details.

### 1.1 Hamiltonian

An atom can be modelled as a collection of charged point particles. The simplest Hamiltonian one can write is therefore

$$H_A = \sum_{\alpha} \frac{\mathbf{p}_{\alpha}^2}{2M_{\alpha}} + \frac{1}{2} \sum_{\alpha \neq \beta} \frac{q_{\alpha} q_{\beta}}{4\pi\epsilon_0 |\mathbf{r}_{\alpha} - \mathbf{r}_{\beta}|} \quad (1.1)$$

where  $\alpha$  labels the particles,  $M_{\alpha}, q_{\alpha}$  are their masses and charges. We try to use in this lecture SI units. (In cgs units, drop the  $4\pi\epsilon_0$ .) The interaction term corresponds to the electrostatic (or Coulomb) field created by the charges.

More advanced atomic models adopt a relativistic viewpoint, take into account the electron spin, the magnetic field created by the motion of the particles, the corresponding spin-orbit interaction, the spin-spin coupling, the hyperfine interaction etc. Theoretical atomic physics computes all these corrections to the energy levels and matrix elements ‘from first principles’. Typically, no simple analytical results can be found for atoms with more than two or three electrons, say.

hydrogen		H	$n$	$l = 0$	$l = 1$	$l = 2$	$\dots$
lithium		Li	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
sodium	Natrium	Na	3	3s	3p	3d	
potassium	Kalium	K	2	2s	2p		
rubidium		Rb	1	1s			
cesium		Cs					
francium		Fr					

Table 1.1: Left: the series of alkaline atoms. Right: Spectroscopic notation for energy levels of hydrogen-like atoms.

For our purposes, it is not useful to go into so much detail. Instead, we use a simplified description of the atom states that captures their essential properties. Good examples are atoms with a single electron in the outer shell (the alkaline series), as listed in table 1.1. Their energy levels are to a good approximation given by a modified Balmer formula

$$E_{nl} = -\frac{e^2}{8\pi\epsilon_0 a_0 (n + \delta_l)^2} = -\frac{\text{Ryd}}{(n + \delta_l)^2}, \quad \begin{array}{l} n = 1, 2, \dots \\ l = 0, \dots, n-1 \end{array} \quad (1.2)$$

The Bohr radius  $a_0 = 4\pi\epsilon_0\hbar^2/me^2 \approx 0.5 \text{ \AA}$  gives the typical size of the electron cloud, and the ‘quantum defect’  $\delta_l$  lifts the degeneracy of the hydrogen levels.<sup>1</sup> The energy scale is given by the Rydberg constant  $\text{Ryd} \approx 13.6 \text{ eV}$ .

The frequency of electromagnetic radiation emitted by atoms is given according to Bohr by

$$\omega_{if} = \frac{E_i - E_f}{\hbar} \quad (1.3)$$

For two typical energy levels  $E_{i,f}$ ,  $\hbar\omega_{if}$  is also of the order of 1 Ryd. If we compute the wavelength of the corresponding electromagnetic radiation, we find

$$\lambda_{if} \sim \frac{2\pi\hbar c}{\text{Ryd}} = \frac{4\pi}{\alpha_{\text{fs}}} a_0 \gg a_0 \quad (1.4)$$

$$\text{fine structure constant } \alpha_{\text{fs}} = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}, \quad (1.5)$$

<sup>1</sup>The charge  $Z|e|$  of the nucleus enters this formula via the quantum defects. In fact, the outer electron ‘sees’ the nucleus screened by the core electrons. This gives a Coulomb potential as for the hydrogen atom, with some modifications due to the core electrons. These are responsible for the lifted degeneracy between the  $l$  states.

which is much longer than the typical size of the atom (the Bohr radius gives the extension of the electronic orbitals). For typical light fields, the atom thus appears like a pointlike object. This property will allow to make the ‘long wavelength approximation’ that simplifies the Hamiltonian for the atom–light interaction.

Another way of looking at the result (1.4) is to interpret the inverse fine structure constant,  $\alpha_{\text{fs}}^{-1} = c(4\pi\epsilon_0\hbar/e^2)$  as the ratio between the speed of light  $c$  and the typical velocity for an electron in the Hydrogen atom (the natural velocity scale in the so-called ‘atomic units’). We see that this velocity is only a few percent of  $c$ , hence we expect that the non-relativistic description we have used so far is a good approximation.

## 1.2 Minimal coupling

According to the rules of electrodynamics, the interaction between a collection of charges with a given electromagnetic field is described by the ‘minimal coupling’ Hamiltonian. This corresponds to the replacement  $\mathbf{p}_\alpha \mapsto \mathbf{p}_\alpha - q_\alpha \mathbf{A}(\mathbf{r}_\alpha, t)$ . In addition, there is the potential energy due to the scalar (electrostatic) potential  $\phi(\mathbf{x}, t)$ , so that we get

$$H_{AF} = \sum_{\alpha} \frac{[\mathbf{p}_\alpha - q_\alpha \mathbf{A}(\mathbf{r}_\alpha, t)]^2}{2M_\alpha} + \frac{1}{2} \sum_{\alpha \neq \beta} \frac{q_\alpha q_\beta}{4\pi\epsilon_0 |\mathbf{r}_\alpha - \mathbf{r}_\beta|} + \sum_{\alpha} q_\alpha \phi(\mathbf{r}_\alpha, t). \quad (1.6)$$

The minimal coupling prescription is related to the freedom of choosing the phase reference of the wave function, as is seen in more detail in the exercises.

**Remark.** This freedom is also called ‘local  $U(1)$  gauge invariance’ because phase factors form the unitary group  $U(1)$ . Local changes in the phase of the wave function generate terms in the Schrödinger equation that can be combined with gauge transformations for the electromagnetic potentials. This connection to the electromagnetic gauge transformations is of great importance for quantum field theory. It allows to construct the coupling to the electromagnetic field from the symmetry properties of the quantum fields. For example, there are theories where electrons and neutrinos are combined into a two-component field, and the interactions are invariant under  $SU(2)$  transformations that mix these two, plus  $U(1)$  transformations of the phase common to the two components. The group  $SU(2) \times U(1)$  is four-dimensional and has four ‘generators’. Each of them corresponds to a vector potential that interacts with the two-component field. In addition to the standard electromagnetic potential (the ‘photon’), there are interactions associated to the ‘massive vector bosons’, called  $W^\pm$  and  $Z^0$ . They convey the ‘weak interaction’ that is responsible for  $\beta$  decay.

When the minimal coupling Hamiltonian (1.6) is expanded to lowest order in the charges  $q_\alpha$ , we obtain the so called ‘ $p \cdot A$ ’ interaction

$$H_{\text{int}} = - \sum_{\alpha} \frac{q_{\alpha}}{2M_{\alpha}} \{ \mathbf{p}_{\alpha} \cdot \mathbf{A}(\mathbf{r}_{\alpha}, t) + \mathbf{A}(\mathbf{r}_{\alpha}, t) \cdot \mathbf{p}_{\alpha} \}. \quad (1.7)$$

In the Coulomb gauge where  $\nabla \cdot \mathbf{A} = 0$ , the ordering of the operators is irrelevant. This interaction is linear in the vector potential, but there is also a second-order (or ‘diamagnetic’) term

$$H_{\text{dia}} = \sum_{\alpha} \frac{q_{\alpha}^2 \mathbf{A}^2(\mathbf{r}_{\alpha}, t)}{2M_{\alpha}} \quad (1.8)$$

When calculations are pushed to second order in the  $p \cdot A$ -coupling, the diamagnetic interaction must be included as well, for consistency. This makes the ‘book-keeping’ in perturbation theory complicated.

A simpler approach is possible, however. We first make the approximation that the field varies slowly on the scale of the displacements  $\mathbf{r}_{\alpha}$  of the charges in the atom. Then we can replace, to lowest order,  $\mathbf{A}(\mathbf{r}_{\alpha}, t) \approx \mathbf{A}(\mathbf{R}, t)$  where  $\mathbf{R}$  is the atomic center of charge. This is called the ‘long-wavelength approximation’ that is well justified for fields near-resonant with typical atomic transitions. Within this approximation, we can find a simpler interaction Hamiltonian that is linear in the electromagnetic field. It is called the ‘ $d \cdot E$ ’ coupling and is strictly linear in the electric field.

$$H_{\text{int}} = -\mathbf{d} \cdot \mathbf{E}(\mathbf{R}, t), \quad (1.9)$$

$$\mathbf{d} = \sum_{\alpha} q_{\alpha}(\mathbf{r}_{\alpha} - \mathbf{R}), \quad (1.10)$$

where  $\mathbf{d}$  is the electric dipole moment of the atom relative to  $\mathbf{R}$ . This version of the  $d \cdot E$  interaction can be derived from the minimal coupling Hamiltonian with a gauge transformation (see the exercises) from the minimal coupling Hamiltonian in the long-wavelength approximation, without invoking an additional approximation.

The advantages of the electric dipole coupling are: the atom couples directly to the field; there is no quadratic interaction term. One must not forget that between the two interactions, the wave function (the atomic state) differs by a unitary transformation. Otherwise, some matrix elements or transition rates may come out differently. This issue is discussed in great detail in Chap. IV of ‘Photons and Atoms – Introduction to Quantum Electrodynamics’ by Cohen-Tannoudji & al. (1987).

### 1.3 Selection rules

Since the electric dipole moment determines the interaction with the light field, a few remarks on its matrix elements are in order. We take as a starting point the basis of the stationary states of an atom, described by the Hamiltonian (1.1). These states are typically described by quantum numbers like parity, angular momentum etc. The ‘selection rules’ specify for which states we know by symmetry that the matrix elements of the electric dipole moment vanish. In that case, the corresponding states are not connected by an ‘electric dipole transition’, or the transition is ‘dipole-forbidden’.

**Parity.** If the state  $|a\rangle$  has a defined parity  $P_a = \pm 1$  (i.e., the electronic wave function  $\psi(\{\mathbf{r}_\alpha\})$  transforms like  $P_a \psi(\{\mathbf{r}_\alpha\})$  when all coordinates are transformed as  $\mathbf{r}_\alpha \mapsto -\mathbf{r}_\alpha$ ), then  $\langle a|\mathbf{d}|a\rangle = 0$ . In addition, the matrix element  $\langle a|\mathbf{d}|b\rangle$  is only nonzero when  $|a\rangle$  and  $|b\rangle$  have different parity.

**Angular momentum.** If there is no electron spin, this is given by  $l$ , and by  $j = l \pm \frac{1}{2}$  for hydrogen-like atoms. The vector operator  $\mathbf{d}$  transforms under rotation like a spin 1 (there are three different basis vectors). One can introduce a basis  $\mathbf{e}_q$  ( $q = -1, 0, 1$ ) that are eigenvectors of  $L_3$  as well and write  $\mathbf{d} = \sum_q d_q \mathbf{e}_q$ . The product  $\mathbf{e}_q|l, m\rangle$  then is an eigenstate of  $L_3$  with eigenvalue  $q+m$ . Therefore, the matrix element with  $|l', m'\rangle$  is only nonzero when  $m' = q + m$ . We find the selection rule

$$|m - m'| \leq 1.$$

In addition, the product states  $\mathbf{e}_q|l, m\rangle$  can be expanded onto eigenstates of  $\mathbf{L}^2$ . The rules for the ‘addition of angular momentum’ imply that only angular momenta  $l' = l - 1, l, l + 1$  occur in this expansion. This gives the selection rule

$$|l - l'| \leq 1.$$

**Total momentum.** An atom that is in a plane wave state regarding its centre-of-mass motion, with momentum  $\mathbf{P}$  receives an additional momentum  $\hbar\mathbf{k}$  when a photon from a plane electromagnetic wave with wave vector  $\mathbf{k}$  is absorbed. The corresponding ‘recoil velocity’  $\hbar\mathbf{k}/M$  is of the order of a few mm/s to a few cm/s for typical atoms. The atomic recoil plays an important role for atom deceleration and cooling with laser light.

## 1.4 Two-level approximation

For the rest of this lecture, it will be sufficient to write the atomic Hamiltonian in the form

$$H_A = \sum_n E_n |n\rangle \langle n|, \quad (1.11)$$

where the states  $|n\rangle$  are the stationary states corresponding to the energy eigenvalue  $E_n$ . But even this form is too complicated: it contains too many terms when dealing with near-resonant laser light. This is the setting we shall focus on here. One can then retain only a few states to describe the atom.

### 1.4.1 Interaction with near-resonant light

The interaction with near-resonant laser light can be described by the Hamiltonian

$$H_{AL}(t) = -\mathbf{d} \cdot (\mathbf{E}(\mathbf{R}) e^{-i\omega_L t} + \text{c.c.}) \quad (1.12)$$

$$= V e^{-i\omega_L t} + \text{h.c.} \quad (1.13)$$

where the electric field is assumed to be monochromatic with (angular) frequency  $\omega_L$ . The term  $\mathbf{E}(\mathbf{R}) e^{-i\omega_L t}$  is called the ‘positive frequency part’ of the field: its time evolution is the same as for a solution of the time-dependent Schrödinger equation (with positive energy  $\hbar\omega_L$ ).

If the field is quantized, Eq.(1.13) applies in similar form in the interaction picture and involves photon annihilation operators  $a_k$  and creation operators. We give the corresponding coupling Hamiltonian in Eq.(1.16) below.

In order to examine what happens to an atom illuminated by such a laser field, we make the ansatz

$$|\psi(t)\rangle = \sum_n c_n(t) e^{-iE_n t/\hbar} \quad (1.14)$$

where the coefficients  $c_n(t)$  become time-dependent because of the interaction. This transformation for the probability amplitudes of the atomic state is called the *interaction picture* with respect to  $H_A$  (1.11). In this picture, the equation of motion for the  $c_n(t)$  is given by

$$i\hbar\dot{c}_n = \sum_m \langle n|H_{AL}(t)|m\rangle e^{-i\omega_{mn}t} c_m(t), \quad (1.15)$$

where  $\hbar\omega_{mn} = E_m - E_n$  is the Bohr frequency.

With a quantized field, the interaction Hamiltonian has to be replaced according to

$$\begin{aligned} \langle n|H_{AL}(t)|m\rangle e^{-i\omega_{mn}t} &\mapsto \langle n|H_{AF}(t)|m\rangle \\ H_{AF}(t) &= -\sum_k \mathcal{E}_k \left( a_k(t) \mathbf{f}_k \cdot \mathbf{d}(t) + a_k^\dagger(t) \mathbf{f}_k^* \cdot \mathbf{d}(t) \right) \end{aligned} \quad (1.16)$$

where  $\mathcal{E}_k = \sqrt{\hbar\omega_k/(2\varepsilon_0V)}$  is the electric field amplitude at the one-photon level and  $\mathbf{f}_k = \mathbf{f}_k(\mathbf{R})$  is the normalized mode function, evaluated at the position  $\mathbf{R}$  of the atom. The photon operators  $a_k$  now carry their ‘free’ time dependence,  $a_k(t) = a_k e^{-i\omega_k t}$ , which is the Heisenberg operator under free evolution. The time dependence of the (freely evolving) dipole operator is given by  $\mathbf{d}(t) = \exp(iH_A t) \mathbf{d} \exp(-iH_A t)$ . The action of the atomic Hamiltonian on the bra/kets  $\langle n|$  and  $|m\rangle$  gives the same exponentials as with a classical field.

Eq.(1.15) is an (infinite-dimensional) linear system of differential equations with time-dependent coefficients. Nothing very difficult, but it is cumbersome to find explicit solutions. So let us resort to perturbation theory. This is called time-dependent perturbation theory in quantum mechanics. Assume that we initially prepared the atom in the state  $|a\rangle$ . This means for the probability amplitudes  $c_n(0) = \delta_{na}$ . To order zero in the atom-light interaction, these amplitudes do not change in time, since we have already separated off the exponentials that appear in the time-dependent Schrödinger equation in Eq.(1.14).

Since the right hand side of (1.15) is already of first order in the interaction, we insert there the zero’th order solution  $c_m(t) = c_m(0) = \delta_{ma}$  and get after one integration

$$c_n(t) = c_n(0) - \frac{i}{\hbar} \int_0^t d\tau \langle n|H_{AL}(\tau)|a\rangle e^{-i\omega_{an}\tau} \quad (1.17)$$

The integral gives for a monochromatic field

$$\begin{aligned} \frac{i}{\hbar} \int_0^t d\tau \langle n|H_{AL}(\tau)|a\rangle e^{-i\omega_{an}\tau} &= \langle n|V|a\rangle \frac{1 - e^{-i(\omega_{an} + \omega_L)t}}{\hbar(\omega_{an} + \omega_L)} + \\ &+ \langle n|V^\dagger|a\rangle \frac{1 - e^{-i(\omega_{an} - \omega_L)t}}{\hbar(\omega_{an} - \omega_L)} \end{aligned} \quad (1.18)$$

The two terms in this result have distinct physical interpretations, related to the denominators.

**Absorption.** The first denominator leads to a ‘large’ result when  $\omega_{an} + \omega_L = 0$ , or  $E_n = E_a + \hbar\omega_L$ . One says that the atom went from the state  $|a\rangle$  to the higher-lying state  $|n\rangle$  by absorbing one ‘energy quantum’ (‘photon’). (Recall that the

amplitude  $c_n$  for the state  $|n\rangle$  is increased in Eq.(1.17).) This process is governed by the ‘positive frequency’ component  $V e^{-i\omega_L t}$  of the interaction Hamiltonian (corresponding to the positive frequency component of the electromagnetic field). In the quantized description of the light field, this component corresponds to a ‘annihilation operator’ that removes one photon from the field.<sup>2</sup> If we fix the states  $|a\rangle$  and  $|n\rangle$  such that the condition for absorption is satisfied, then the second term in Eq.(1.18) has a ‘large’ denominator,  $\omega_{an} - \omega_L \approx -2\omega_L$ . This term is therefore much smaller than the first one. This suggests that this second term could be neglected. This approximation is called the ‘resonance approximation’ or the ‘rotating wave approximation’ (an admittedly strange name). If we keep the non-resonant term, we deal in the quantum theory with a ‘virtual’ process where the atom passes into a state with a higher energy and at the same time, a photon is created.

**Emission.** The second term in (1.18) corresponds to a transition with the energy balance  $E_n = E_a - \hbar\omega_L$ : the atom makes a transition to a lower-lying state. In the quantized field description, a ‘photon’ is created (by the creation operator  $a_k^\dagger$  in the expansion of the field operator).

To summarize, in the two-state approximation, we only retain those two levels  $|a\rangle$  and  $|n\rangle$  which show the smallest resonance denominator  $\omega_{na} \pm \omega_L$ , and discard the others. This is possible for a ‘detuning’  $\Delta = \omega_L - |\omega_{na}|$  small compared to the typical differences between atomic transition frequencies. This condition is easily achieved since the frequency differences are of the order of 1 eV, and this is a ‘huge’ detuning to drive an atomic transition.

**Remark.** The description of absorption and emission, as we encounter it here, does not explicitly require the quantization of the light field. These processes also occur in a ‘classical’ time-dependent potential because energy is not conserved there, as is well known in classical mechanics. One can push the analogy even further: a weak monochromatic excitation of a mechanical system reveals the system’s ‘resonance frequencies’. For an atom, these are apparently given by the Bohr frequencies. The only difference to a mechanical system is that we are inclined to use different names for the excitations with positive and negative frequencies, since in the atomic energy spectrum, there is a definite difference between ‘going up’ and ‘going down’ (there exists a ground state).

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<sup>2</sup>It is a simple exercise to show that annihilation operators, i.e., operators whose application gives energy eigenstates with a lower energy, evolve at positive frequencies in the Heisenberg picture.

## 1.4.2 Two-level language

Summarizing, we can retain for near-resonant interactions only two atomic levels. The advantage is that one can get much more analytical results in this simpler setting. In the following, we introduce some notations for two-level atoms that are often used (and will be used in the rest of this lecture).

**Atomic Hamiltonian.** The standard notation for the two states is  $|g\rangle$  for the ground state and  $|e\rangle$  for the excited state. The Bohr frequency is often written  $\omega_A = \omega_e - \omega_g > 0$ . The atomic energy levels are often referenced to a zero energy lying between both states, this gives:

$$H_A = \frac{\hbar\omega_A}{2}|e\rangle\langle e| - \frac{\hbar\omega_A}{2}|g\rangle\langle g| \quad (1.19)$$

It is also useful to identify the two-dimensional Hilbert space of the two-level atom with the  $\mathbb{C}^2$ , using the basis vectors  $(1, 0)^T \leftrightarrow |e\rangle$  and  $(0, 1)^T \leftrightarrow |g\rangle$ . The Hamiltonian then becomes the diagonal matrix

$$H_A = \frac{\hbar\omega_A}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar\omega_A}{2} \sigma_3 \quad (1.20)$$

where  $\sigma_3$  is the third Pauli matrix. Indeed, it is obvious that a two-dimensional Hilbert space can be identified with the Hilbert space of a spin 1/2.

**Interaction Hamiltonian.** The spin analogy can even be pushed further, and now the Pauli matrices  $\sigma_1$  and  $\sigma_2$  come into play. These matrices do have nonzero matrix elements between different states, for example  $\langle g|\sigma_1|e\rangle = 1$ , just as the coupling Hamiltonian  $V(t)$ . (If  $\langle g|V(t)|e\rangle = 0$ , it would not be very useful to concentrate on just the states  $|g\rangle$  and  $|e\rangle$ .) Let us introduce the complex quantities  $\Omega$ ,  $\Omega'$  ( $\Omega$  is the so-called ‘Rabi frequency’) by writing

$$\langle e|V(t)|g\rangle = \langle e|V|g\rangle e^{-i\omega_L t} + \langle e|V^\dagger|g\rangle e^{i\omega_L t} \equiv \frac{\hbar\Omega}{2} e^{-i\omega_L t} + \frac{\hbar\Omega'}{2} e^{i\omega_L t} \quad (1.21)$$

where  $V$  is the time-independent complex operator coming with the positive frequency field (cf. the first equal sign).

The diagonal matrix elements of the electric dipole interaction vanish if the states  $|g\rangle$  and  $|e\rangle$  have a definite parity. So we can express the interaction Hamiltonian by the following two matrices

$$H_{\text{int}} = \frac{\hbar}{2} \begin{pmatrix} 0 & \Omega e^{-i\omega_L t} \\ \Omega^* e^{i\omega_L t} & 0 \end{pmatrix} + \frac{\hbar}{2} \begin{pmatrix} 0 & \Omega' e^{i\omega_L t} \\ \Omega'^* e^{-i\omega_L t} & 0 \end{pmatrix} \quad (1.22)$$

We have used the fact that this matrix must be hermitian (since it represents the observable ‘interaction energy’) to fix the lower left matrix element.

For a quantized field, the electric field operator occurs in the off-diagonal elements of these matrices, projected onto the vector  $\mathbf{d}_{eg} = \langle e|\mathbf{d}|g\rangle$  of dipole matrix elements.

**Rotating wave approximation.** The interaction Hamiltonian (1.22) is still too complicated, and even inconsistent with the two-level approximation we just made. This can be seen by working out again the time-dependent perturbation theory for this two-state system. For definiteness, let us start with an atom in the ground state. One finds that the positive frequency terms involving  $\Omega$  or  $\langle e|V|g\rangle$  come with a resonant denominator  $1/(\omega_A - \omega_L)$ , while the negative frequency terms involving  $\Omega'$  are proportional to  $1/(\omega_A + \omega_L)$  where the denominator never goes through zero. Since we discarded this kind of ‘nonresonant’ terms when we focussed on two levels, we discard them here as well for consistency. This approximation is called the ‘rotating wave approximation’ (r.w.a.) which is a weird name that we comment on below. In the rest of this lecture, we shall make the r.w.a. as often as possible (it must be avoided when energy level shifts are computed, e.g.).

In the r.w.a., the atomic Hamiltonian thus becomes the matrix

$$H_{AL} = \frac{\hbar\omega_A}{2}\sigma_3 + \frac{\hbar}{2} \begin{pmatrix} 0 & \Omega e^{-i\omega_L t} \\ \Omega^* e^{i\omega_L t} & 0 \end{pmatrix} \quad (1.23)$$

$$= \frac{\hbar\omega_A}{2}\sigma_3 + \frac{\hbar}{2} (\Omega^* e^{i\omega_L t} \sigma + \sigma^\dagger \Omega e^{-i\omega_L t}) \quad (1.24)$$

$$= \frac{\hbar\omega_A}{2}\sigma_3 + \frac{\hbar\Omega}{2} (\sigma_1 \cos \omega_L t + \sigma_2 \sin \omega_L t) \quad (1.25)$$

In the second line, we have used the two-level annihilation ( $\sigma = |g\rangle\langle e|$ ) and creation ( $\sigma^\dagger = |e\rangle\langle g|$ ) operators. We can always choose the origin of time such that the Rabi frequency  $\Omega$  is real. Recall the link to the vector  $\mathbf{d}_{eg} = \langle e|\mathbf{d}|g\rangle$  with the matrix elements of the dipole operator:  $\hbar\Omega = -2\mathbf{d} \cdot \mathbf{E}(\mathbf{r}_A)$  where  $\mathbf{E}(\mathbf{r})$  is the positive frequency amplitude of the electric field at the atom’s position.

With a quantized field, add the Hamiltonian  $H_F$  for the field and make the replacement

$$\frac{\hbar\Omega}{2} e^{-i\omega_L t} \mapsto - \sum_k \mathcal{E}_k a_k \mathbf{f}_k \cdot \mathbf{d}_{eg} \quad (1.26)$$

One can also use an operator-valued Rabi frequency per mode,  $\Omega_k$  in the following that can be read off from the summand in Eq.(1.26).

### 1.4.3 Overview

The Hamiltonian (1.24), combined with its quantized field formulation, is the basis for most of the phenomena that have been studied in the quantum optics of two-level system. One can discuss the following topics (we give a selection in this lecture):

- Rabi oscillations in a classical monochromatic field;
- spontaneous decay of an excited atom into the continuum of vacuum field modes (initially in the ground state);
- collapse and revival of Rabi oscillations when the atom couples to a single quantized field mode. The collapse and the revival occurs because the Rabi frequency depends on the photon number, and the oscillations for the different Fock state components of a field state (a coherent state, for example) get out of phase;
- resonance fluorescence is the radiation emitted by an atom driven by a near-resonant laser field. This combines Rabi oscillations in a classical field with the emission of photons into the empty mode continuum. Of particular interest is the spectrum of this emission: it contains, for sufficiently strong driving, two sidebands, split by the Rabi frequency from the central line (centered at the laser frequency). The central line contains a monochromatic component ('elastic scattering', related to the laser-induced dipole moment as in classical electrodynamics) and a broadened component of Lorentzian shape, related to spontaneous emission. This spectrum is a cornerstone of quantum optics and one of the few examples of a non-perturbative calculation in quantum electrodynamics.

## 1.5 States of the two-level atom

### 1.5.1 Spin 1/2 analogy

We now come back to the spin 1/2 analogy. The Hamiltonian (1.25) with the atomic energies and the atom–laser interaction has the same form as the Hamiltonian for a spin 1/2 in a time-dependent magnetic field,

$$H_{\text{spin}} = \boldsymbol{\sigma} \cdot \mathbf{B}(t) \quad (1.27)$$

where  $\sigma$  is the vector of Pauli matrices and the ‘magnetic field’  $\mathbf{B}(t)$  actually has the dimensions of an energy (we took a unity magnetic moment). The magnetic field rotates at the laser frequency around the  $x_3$ -axis:

$$\mathbf{B}(t) = \frac{\hbar}{2} \begin{pmatrix} \Omega \cos \omega_L t \\ \Omega \sin \omega_L t \\ \omega_A \end{pmatrix} \quad (1.28)$$

It is useful to change the coordinate frame such that it co-rotates with this field (this is the ‘rotating frame’). In this frame, the ‘effective magnetic field’ is static<sup>3</sup>,

$$\mathbf{B}_{\text{eff}} = \frac{\hbar}{2} \begin{pmatrix} \Omega \\ 0 \\ \omega_A \end{pmatrix}. \quad (1.29)$$

The transformation into the rotating frame also changes the wave function of our two-state particle by a unitary transformation

$$U(t) = \exp\{-i\omega_L t \sigma_3 / 2\} = \begin{pmatrix} e^{-i\omega_L t / 2} & 0 \\ 0 & e^{i\omega_L t / 2} \end{pmatrix} \quad (1.30)$$

We observe that this is the transformation we used in Eq.(1.36) to go into the interaction picture (on resonance where  $\omega_L = \omega_A$ ). This unitary transformation being time-dependent, we get also a modification of the Hamiltonian proportional to  $-i\hbar U^\dagger \partial_t U = -\hbar\omega_L \sigma_3$ . All told, we find the Hamiltonian *in the rotating frame*

$$H = -\frac{\hbar\Delta}{2}\sigma_3 + \frac{\hbar\Omega}{2}\sigma_1 \quad (1.31)$$

where the *detuning* is given by the difference between the laser frequency and the atomic transition frequency

$$\Delta = \omega_L - \omega_A. \quad (1.32)$$

Note that the laser and atomic frequencies have disappeared from the Hamiltonian and only their difference (the detuning) occurs. As a consequence, the relevant time scales (given by  $1/\Delta$  and  $1/\Omega$ ) are typically much longer than the optical period  $2\pi/\omega_L$ . On these long time scales, nonresonant processes remain ‘virtual’ and cannot be directly observed. This is consistent with the neglect of nonresonant levels (two-state approximation) and of the nonresonant two-state coupling (rotating wave approximation).

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<sup>3</sup>If we had kept the nonresonant terms in the Hamiltonian, the magnetic field would also show a time-dependent component rotating at the frequency  $2\omega_L$ .

## 1.5.2 The Bloch sphere

The language of a spin 1/2 is often used to visualize the dynamics of a two-level atom. Let us compute the components of the average spin vector  $(u, v, w)^T \equiv \langle \boldsymbol{\sigma} \rangle$  (the ‘Bloch vector’) in the state  $|\psi(t)\rangle = (c_e(t), c_g(t))^T$ . Its third component is related to the occupation probabilities  $|c_e(t)|^2, |c_g(t)|^2$  (the ‘populations’): it gives the *inversion*, i.e., the difference of the ground and excited state populations

$$w \equiv \langle \sigma_3 \rangle = |c_e(t)|^2 - |c_g(t)|^2. \quad (1.33)$$

In the ground state, one has  $\langle \sigma_3 \rangle = -1$ . The other two components are given by

$$u \equiv \langle \sigma_1 \rangle = 2 \operatorname{Re}(c_e^*(t)c_g(t)), \quad (1.34)$$

$$v \equiv \langle \sigma_2 \rangle = 2 \operatorname{Im}(c_e^*(t)c_g(t)). \quad (1.35)$$

They are only nonzero when the atom is in a superposition of the ground and excited states.

In the exercises, you show that the Bloch vector is of unit length. It can thus be represented by points on the surface of a sphere, the ‘Bloch sphere’ (see Fig.1.1). The ‘south pole’ of the sphere corresponds to an atom in the ground state, while the atom is in the excited state at the ‘north pole’. In between these two, the atom is in a superposition of ground and excited states. In particular, around the equator, both ground and excited state are occupied with probability 1/2 (the inversion is zero). The longitude along the equator is fixed by the phase difference between ground and excited state.

If no external laser field is applied, the Bloch vector rotates at the frequency  $\Delta$  counterclockwise around the vertical axis. In particular, the occupation probabilities do not change with time: the inversion, being the projection onto the vertical axis, is constant. (One may think that in this case, the transformation into the rotating frame is of limited use. If you do not apply it, you find a rotation at the atomic transition frequency  $\omega_A$ .) With the laser on, the rotation axis gets tilted, and for example, an atom initially in the ground state develops an excited component. This is one of the examples of the next subsection

## 1.5.3 Mixed states

Density operator for a two-level system.

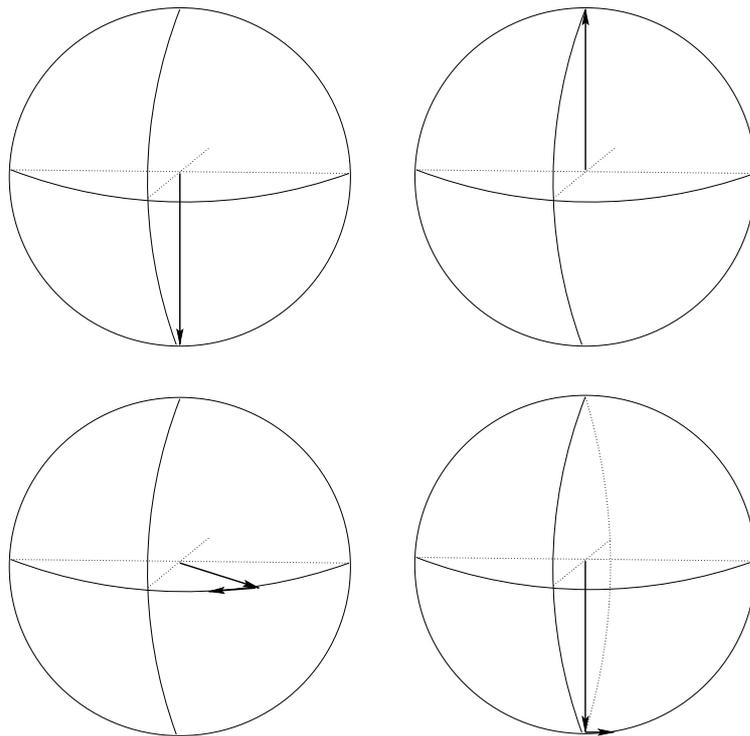


Figure 1.1: Upper left: Bloch vector for an atom in the ground state. Upper right: excited state. Lower left: superposition of ground and excited states with equal weight and free rotation without a laser field. Lower right: start of resonant Rabi oscillations with an initial ground state. The  $x_1$ -axis points to the right, the  $x_3$ -axis upwards.

## 1.6 Two-level dynamics

### 1.6.1 Rabi oscillations

The most simple case of atom-laser dynamics is a laser ‘on resonance’, i.e.,  $\omega_L = \omega_A$ . If we change again into an interaction picture,

$$|\psi(t)\rangle = \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix} \begin{pmatrix} c_e(t) \\ c_g(t) \end{pmatrix}, \quad (1.36)$$

we find similar to Eq.(1.15)

$$i\hbar\dot{c}_e = \frac{\hbar\Omega}{2}c_g \quad (1.37)$$

$$i\hbar\dot{c}_g = \frac{\hbar\Omega}{2}c_e. \quad (1.38)$$

With the initial conditions  $c_g(0) = 1$ ,  $c_e(0) = 0$ , the solution is

$$\dot{c}_e = -i \sin(\Omega t/2) \quad (1.39)$$

$$c_g = \cos(\Omega t/2). \quad (1.40)$$

The excited state probability thus oscillates between 0 and 1 at a frequency  $\Omega/2$ . This phenomenon is called ‘Rabi flopping’. It differs from what one would guess from ordinary time-dependent perturbation theory where one typically gets linearly increasing probabilities. That framework, however, applies only if the final state of the transition lies in a continuum which is not the case here. Rabi flopping also generalizes the perturbative result (1.17) which would give a quadratic increase  $|c_e|^2 \propto t^2$  that cannot continue for long times. But instead of saturating, the atomic population returns to the ground state.

Every experimentalist is very happy when s/he observes Rabi oscillations. It means that any dissipative processes have been controlled so that they happen at a slower rate. In a realistic setting, one gets a damping of the oscillation amplitude towards equilibrium populations.

**Rabi pulses.** Rabi oscillations with a fixed interaction time are often used to implement coherent operations on an atom or spin. The corresponding evolution operator is given by (we focus on the resonant case)

$$U_\theta = \exp\{-i\theta\sigma_1/2\} = \cos(\theta/2) - i\sigma_1 \sin(\theta/2) \quad (1.41)$$

with  $\theta = \Omega t$ . After one cycle of Rabi oscillations,  $\Omega t = 2\pi$  (a ‘ $2\pi$ -pulse’), the atom returns to its ground state — but its wave function has changed sign. This sign change is well-known from spin 1/2 particles: the corresponding unitary transformation reads

$$U_{2\pi} = \cos(\pi) - i\sigma_1 \sin(\pi) = -1 \quad (1.42)$$

A more interesting manipulation is a ‘ $\pi$ -pulse’,  $\Omega t = \pi$ , that flips the ground and excited state:

$$U_{\pi} = \cos(\pi/2) - i\sigma_1 \sin(\pi/2) = -i\sigma_1 \quad (1.43)$$

Finally, a ‘ $\pi/2$ -pulse’ takes the atom into a superposition of ground and excited states with equal weight (a Bloch vector on the equator of the Bloch sphere)

$$U_{\pi/2} = \cos(\pi/4) - i\sigma_1 \sin(\pi/4) = \frac{1 - i\sigma_1}{\sqrt{2}}$$

$$U_{\pi/2}|g\rangle = \frac{1}{\sqrt{2}}|g\rangle - \frac{i}{\sqrt{2}}|e\rangle$$

If the laser is shut off after such a pulse, the Bloch vector will continue to rotate along the equator at the frequency  $\Delta$ .

## 1.6.2 Collapse and revival

If the light field is described as a single quantized mode, an additional feature occurs in the Rabi oscillations. The key point is that the coupling Hamiltonian,  $g(a^\dagger\sigma + \sigma^\dagger a)$ , now couples the states  $|g, n\rangle$  and  $|e, n-1\rangle$  where  $n$  is the photon number. These states are split (on resonance) in energy by the ‘Rabi splitting’  $g\sqrt{n}$ . Recall that this splitting was  $|\Omega|$  for a classical laser field, proportional to the field amplitude. This is mimicked by the scaling with  $\sqrt{n}$  since the photon number  $n$  is proportional to the field intensity.

In each sub-space spanned by  $|g, n\rangle$  and  $|e, n-1\rangle$ , the system thus performs Rabi oscillations with a slightly different frequency. If one starts with a coherent state  $|\alpha\rangle$  for the field mode, the Rabi oscillations will thus evolve at a mean frequency  $\approx g|\alpha|$ , but at large times, the oscillations will ‘get out of phase’. This leads to a ‘collapse’ of the Rabi oscillation amplitude, as illustrated in Figure 1.2. It happens on the time scale  $1/g$  which is a factor  $|\alpha|$  times longer than the period of the initial Rabi oscillations. At still larger times, of order  $1/(g|\alpha|)$ , the amplitude of the oscillations ‘revives’ again. This is due to the fact that the Rabi frequencies form a discrete set. A more detailed analysis is presented in Sec.??.

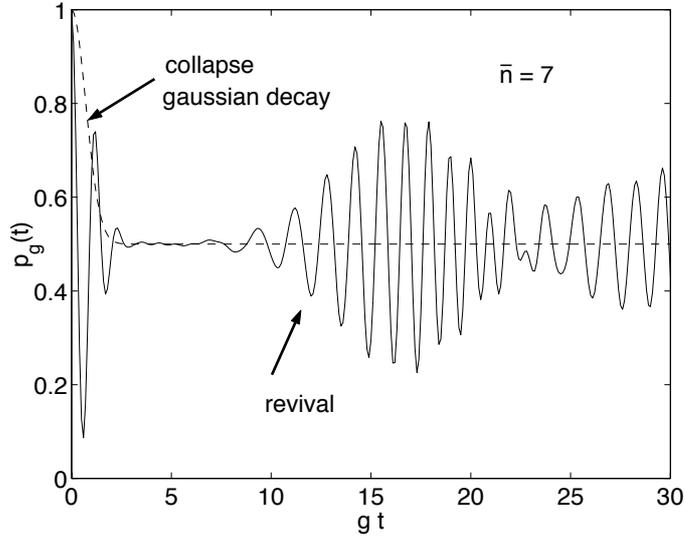


Figure 1.2: Ground state occupation  $p_g(t)$  for a two-level atom coupled to a single mode, initially in the coherent state  $|\alpha\rangle$  with  $|\alpha|^2 = 7$ . Time is in units of the ‘single-photon Rabi frequency’  $g$ .

### 1.6.3 Spontaneous decay

We now describe how the excited states of the two-level atom decay by emitting a photon into an ‘empty’ mode of the electromagnetic field. This phenomenon can conveniently be described by the equations of ‘radioactive decay’ (a pair of ‘rate equations’)

$$\frac{dp_e}{dt} = -\gamma p_e, \quad \frac{dp_g}{dt} = +\gamma p_g \quad (1.44)$$

The rate  $\gamma$  gives the probability per unit time of emitting a photon and putting the atomic population from the excited state down to the ground state. The total population is conserved, as it should be for a process where the atom just changes its internal state. (In radioactive decay, ‘e’ would be a plutonium and ‘g’ an uranium atom, and the ‘photon’ an  $\alpha$ -particle.) In addition, we need equations for the decay of off-diagonal elements of the density matrix like  $\rho_{eg}$ . These equations cannot be chosen arbitrarily because we require that the density operator  $\rho$  remains positive under time evolution. We discuss this in more detail later in the lecture.

To compute the rate  $\gamma$ , we need Fermi’s Golden Rule, a standard result from time-dependent perturbation theory. We derive this after we have learned about the quantization of the electromagnetic field.

# Bibliography

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