Einführung in die Quantenoptik I

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Problem 2.1 – Two-level language (12 points)

(i) A typical quantum state of a two-level system can be written in the form

$$|\psi\rangle = c_{\rm g}|{\rm g}\rangle + c_{\rm e}|{\rm e}\rangle = \begin{pmatrix} c_e \\ c_g \end{pmatrix} \tag{2.1}$$

Explain the meaning of the symbols in the two expressions. Write down the Dirac kets $|{\rm g}\rangle$, $|{\rm e}\rangle$ as two-component vectors. Write down the "normalization condition" for this state.

(ii) Explain the following statement: "For a two-level system, physical observables are represented by hermitean 2×2-matrices." Consider the following observable

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = |{\rm e}\rangle\langle{\rm e}| - |{\rm g}\rangle\langle{\rm g}| \tag{2.2}$$

and check that the two expressions (matrix and ket-bra notation) are consistent by working out the expectation value $\langle\sigma_3\rangle = \langle\psi|\sigma_3|\psi\rangle$. Why is the name "inversion" consistent with the physical interpretation of this observable?

(iii) Repeat the previous exercise for the "observable"

$$\sigma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = |{\rm g}\rangle\langle{\rm e}| \tag{2.3}$$

and its hermitean conjugate $\sigma^\dagger$. Questions for the physical interpretation: why are these only "observables" in quotes? What do the names "dipole operator" and "two-state annihilation (creation) operator" try to tell you? Give examples for quantum states with $\langle\sigma\rangle = 0$ and $\neq 0$.

(iv) A two-level system coupled to a laser can be described by the following Hamiltonian (some approximations and simplifications have been made here)

$$H = -\frac{\hbar\Delta}{2}\sigma_3 + \frac{\hbar}{2}(\Omega^*\sigma + \Omega\sigma^\dagger) \tag{2.4}$$

where Δ and Ω are frequencies. (Ω can be complex.) Write down a ket-bra expression and a 2×2 matrix for H in the same basis as in (i–iii) and show that the amplitudes $c_{\rm g}$, $c_{\rm e}$ satisfy the following time-dependent Schrödinger equation

$$\begin{aligned} \mathrm{i}\dot{c}_{\rm e} &= -\frac{\Delta}{2} c_{\rm e} + \frac{\Omega}{2} c_{\rm g} \\ \mathrm{i}\dot{c}_{\rm g} &= \frac{\Delta}{2} c_{\rm g} + \frac{\Omega^*}{2} c_{\rm e} \end{aligned} \tag{2.5}$$

Why do you expect that the solution to these equations involves the frequencies $\pm\sqrt{\Delta^2 + |\Omega|^2}$ (called generalized Rabi frequency)? Solve the equations for the "resonant case" $\Delta = 0$.

(v) In the Heisenberg picture, the operators σ_3, σ, and $\sigma^\dagger$ are time-dependent. Derive the following differential equations

$$\dot{\sigma}_3 = \mathrm{i}(\Omega^* \sigma - \Omega \sigma^\dagger) \tag{2.6}$$

$$\dot{\sigma} = \mathrm{i}\Delta\sigma + \mathrm{i}\frac{\Omega}{2}\sigma_3 \tag{2.7}$$

$$\dot{\sigma}^\dagger = -\mathrm{i}\Delta\sigma^\dagger - \mathrm{i}\frac{\Omega^*}{2}\sigma_3 \tag{2.8}$$

Solve the equations for the "free atom" $\Omega = 0$.

Problem 2.2 – Quantum states of a two-level system ("qubit") (8 points)

The density operator ρ for a two-level system is a hermitean 2×2 matrix with trace one. Is ρ an observable? The expectation value of a system operator A is given by $\langle A \rangle = \langle A \rangle_\rho = \mathrm{tr}(A\rho)$ where tr is the trace.

(i) Show that this rule gives the same expectation values as the "standard prescription" for the quantum state $|\psi\rangle$ given in Eq.(2.1), when the density operator is defined as $\rho = |\psi\rangle\langle\psi|$.

(ii) Consider a density matrix of the form

$$\rho = \frac{1}{2}\left(\mathbb{1} + \sum_{j=1}^{3} s_j \sigma_j \right). \tag{2.9}$$

Show that the real numbers s_j (the components of the so-called *Bloch vector*) are equal to the expectation value of the operators σ_j. Here, $\sigma_1 = \sigma + \sigma^\dagger$, $\sigma_2 = \mathrm{i}(\sigma - \sigma^\dagger)$, and σ_3 are the Pauli matrices.

(ii) The expectation value of the density matrix itself is given by $P = \mathrm{tr}\,\rho^2$. This real number is called the *purity* and gives us a way to distinguish between "pure states" and "mixed states". A quantum state is called *pure* if $P = 1$. Conclude that the density matrix of a pure state can be written as $\rho = |\psi\rangle\langle\psi|$, whatever the dimension of the system.

(iii) Show that the two-level system is in a pure state if and only if the Bloch vector is on the unit sphere: $\sum_j s_j^2 = 1$.