Einführung in die Quantenoptik I

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Problem 3.1 – Thomas-Reiche-Kuhn sum rule (5 points)

(i) In the lecture, we have started to work with the dipole operator that takes the form $\mathbf{d} = e\mathbf{x}$ for an atom with a single electron. Express the matrix elements of the momentum operator $\mathbf{p}$ between energy eigenstates in terms of the dipole operator.

(ii) Prove the Thomas-Reiche-Kuhn sum rule:

$$\sum_b \omega_{ba} |\langle b|\mathbf{d}|a\rangle|^2 = \frac{\hbar e^2}{2m} \tag{3.1}$$

where $|a\rangle$, $|b\rangle$ are energy eigenstates and $\hbar\omega_{ba} = E_b - E_a$ are the Bohr frequencies. Why does this sum rule illustrates the limited validity of the two-level atom model?

Hint. Work out commutators between $\mathbf{x}$, $\mathbf{p}$, and the Hamilton operator of the atom. You may neglect the laser field in this problem.

Problem 3.2 – Atomic polarizability (7 points)

(i) In the lecture, we are going to use time-dependent perturbation theory for a two-level atom driven by a monochromatic laser field, $\mathbf{E}(t) = \mathbf{E}\,\mathrm{e}^{-\mathrm{i}\omega_L t} + \mathrm{h.c.}$. This gives an approximation to the state vector $|\psi(t)\rangle$ and one may ask for the average (induced) dipole moment

$$\langle\psi(t)|\left(\mathbf{d}_{\mathrm{ge}}\sigma + \mathrm{h.c.}\right)|\psi(t)\rangle = \alpha(\omega_L)\mathbf{E}\,\mathrm{e}^{-\mathrm{i}\omega_L t} + \mathrm{h.c.} \tag{3.2}$$

The coefficient of proportionality in Eq.(3.2) is called the atomic polarizability. It depends on the initial state of the atom and is in general a tensor. If the atom is initially in its ground state, the resulting polarizability is

$$\alpha_g(\omega) = \frac{(2\omega_{\mathrm{eg}}/\hbar)\mathbf{d}_{\mathrm{ge}} \otimes \mathbf{d}^*_{\mathrm{ge}}}{\omega^2_{\mathrm{eg}} - \omega^2} \tag{3.3}$$

(So far, you have nothing to calculate.) Putting together the information from the lecture, one finds that the Bloch vector of a two-level atom $\mathbf{s} = (s + s^*, \mathrm{i}(s - s^*), s_3)$ (the expectation value of the spin vector $\boldsymbol{\sigma}$) satisfies the equations

$$\frac{\mathrm{d}}{\mathrm{d}t}s = -\left(\mathrm{i}\omega_A + \Gamma\right)s + \mathrm{i}(\Omega/2)\,\mathrm{e}^{-\mathrm{i}\omega_L t}\,s_3 \tag{3.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}s_3 = -\gamma(s_3 + 1) + \mathrm{i}\left(\Omega^*\,\mathrm{e}^{\mathrm{i}\omega_L t}s - \Omega\,\mathrm{e}^{-\mathrm{i}\omega_L t}s^*\right) \tag{3.5}$$

where Γ and γ are phenomenological decay rates and the complex Rabi frequency is given by $-\frac{1}{2}\hbar\Omega = \mathbf{d}_{ge}^{*} \cdot \mathbf{E}$.

Now come your tasks: (i) Argue that a steady state solution to the Bloch equations (3.4, 3.5) is of the form $s(t) = s_{\rm stat}\,{\rm e}^{-{\rm i}\omega_L t}$ and $s_3 = s_{3,\rm stat} = {\rm const.}$ (ii) Calculate the stationary amplitudes $s_{\rm stat}$, $s_{3,\rm stat}$ and show in particular that the average "dipole" $s_{\rm stat}$ is given by

$$s_{\rm stat} = \frac{(\Omega/2)\gamma(\Delta - {\rm i}\Gamma)}{\gamma(\Delta^2 + \Gamma^2) + \Gamma|\Omega|^2} \tag{3.6}$$

(iii) Comment on the similarities and differences to Eq.(3.3).

Problem 3.3 – Negative probabilities? (8 points)

When deriving the equations of motion for open quantum systems, one sometimes faces the technical problem that the density operator ρ does not remain a positive operator at all times. Recall that this implies a violation of the inequality

$$\langle\psi|\rho(t)|\psi\rangle \ge 0 \tag{3.7}$$

for some state $|\psi\rangle$ and/or some time t. This problem gives a very simple example to illustrate this issue.

Show that the density operator has a negative eigenvalue if and only if the Bloch vector s has a length $|\mathrm{s}| > 1$ ("one leaves the Bloch sphere"). (Hint: write the matrix elements of ρ in terms of the s_i and compute the determinant.)

Consider the master equation[1] [recall that $s = (s_1 - {\rm i}s_2)/2$]

$$\begin{aligned} \dot{s}_1 &= -\delta s_2 &(3.8)\\ \dot{s}_2 &= (\delta + \delta\omega)s_1 - \gamma s_2 &(3.9)\\ \dot{s}_3 &= -(1+A)\Gamma s_3 + (1-A)\Gamma &(3.10)\end{aligned}$$

where γ and Γ are positive rates, $A \ge 0$, and δ and $\delta\omega$ are real. *Comment* on the differences with respect to the Bloch equations derived in the lecture.

Show that these coefficients must satisfy the following (in)equalities, otherwise there are initial conditions that leave the Bloch sphere[2] (5 bonus points)

$$\begin{aligned} \gamma &= (1+A)\Gamma &(3.11)\\ (1+A)^2\Gamma^2 &\ge \gamma^2 + \delta\omega^2 + (1-A)^2\Gamma^2 &(3.12)\end{aligned}$$

Show that both constraints are satisfied only when $A = 1$ and $\delta\omega = 0$.

Work out the stationary solution to the master equations (3.8–3.10). *Relate* A to an effective temperature of the two-level system and *comment* on the constraint $A = 1$.

Note. This master equation is an attempt to go beyond the rotating wave approximation. Most of these attempts have difficulties at some point (negative probabilities, non-thermal stationary states, violation of fundamental symmetries).

[1]Discussed in Suarez & al, *J. Chem. Phys.* (1992)].

[2]Alicki and Lendi, *Quantum Dynamical Semigroups and Applications*, Springer 1987).