Einführung in die Quantenoptik I

Wintersemester 2010/11
Carsten Henkel

Übungsaufgaben Blatt 6 Ausgabe: 04. Januar 2011
Abgabe: 18. Januar 2011

Problem 6.1 – Commutator of fields (5 points)

In the lecture, we have found that the commutator between the quantized (transverse) vector potential $\mathbf{A}$ and its conjugate momentum field $\mathbf{\Pi}$ should be

$$[A_i(\mathbf{x}), \Pi_j(\mathbf{x}')] = \mathrm{i}\hbar\delta^{\perp}_{ij}(\mathbf{x} - \mathbf{x}') \tag{6.1}$$

(i) Take the rotation with respect to $\mathbf{x}$ and the index i and observe that this is nonzero. Show that one gets the so-called Pauli commutator between the (transverse) fields $\mathbf{E}$ and $\mathbf{B}$:

$$[B_i(\mathbf{x}), E_j(\mathbf{x}')] = \frac{\mathrm{i}\hbar}{\varepsilon_0}\epsilon_{ijk}\frac{\partial}{\partial x_k}\delta(\mathbf{x} - \mathbf{x}') \tag{6.2}$$

(ii) In a mathematically careful (quantum) field theory, the fields $\mathbf{A}$, $\mathbf{E}$, $\mathbf{B}$ etc are actually (operator-valued) distributions that have to be 'smeared out' with suitable smooth test functions $\mathbf{f}(\mathbf{x})$, $\mathbf{g}(\mathbf{x})$. This gives operators that behave in a less singular way; in particular you can multiply fields evaluated at the same point. Consider thus the observables

$$\mathcal{E} = \sqrt{\varepsilon_0}\int \mathrm{d}^3x\, \mathbf{f}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}), \qquad \mathcal{B} = \frac{1}{\sqrt{\mu_0}}\int \mathrm{d}^3x\, \mathbf{g}(\mathbf{x}) \cdot \mathbf{B}(\mathbf{x}) \tag{6.3}$$

Work out the commutator $[\mathcal{E}, \mathcal{B}]$ and conclude that orthogonal components of $\mathbf{E}$ and $\mathbf{B}$ at neighboring points cannot be measured simultaneously. Derive, as in the quantum mechanics I course, the uncertainty relation between the variances $(\Delta\mathcal{E})^2$, $(\Delta\mathcal{B})^2$.

(iii) In the construction of (ii), focus on mode functions of the form $\nabla \times \mathbf{g} = (\omega/c)\mathbf{f}$ with $\mathbf{f}$ being normalized as $\int \mathrm{d}^3x\, \mathbf{f}^2(\mathbf{x}) = 1$ and ω a constant. Consider a quantum state for the field where the average values of $\mathcal{E}$, $\mathcal{B}$ are zero and their variances are identical. What do you get for the 'energy' $\frac{1}{2}\langle\mathcal{E}^2\rangle + \frac{1}{2}\langle\mathcal{B}^2\rangle$?

Interpretation. One 'photon energy' $\hbar|\omega|$ is spread over the spatial support of $\mathbf{f}(\mathbf{x})$.

Problem 6.2 – Photon spin (5 points)

Make yourself acquainted with the field theory version of Emmy Noether's theorem[5]. Understand how the spin angular momentum of the electromagnetic field arises as

$$\mathbf{S} = \int d^3x \, \mathbf{A}_\perp(\mathbf{x}) \times \mathbf{\Pi}(\mathbf{x}). \tag{6.4}$$

i) By correspondence, the operator for the spin angular momentum reads

$$\hat{\mathbf{S}} = \frac{1}{2} \int d^3x \, \left[\hat{\mathbf{A}}_\perp(\mathbf{x}) \times \hat{\mathbf{\Pi}}(\mathbf{x}) - \hat{\mathbf{\Pi}}(\mathbf{x}) \times \hat{\mathbf{A}}_\perp(\mathbf{x}) \right] \tag{6.5}$$

Is the operator ordering really relevant here? Use the canonical field quantization to find a mode expansion of $\hat{\mathbf{S}}$.

ii) Now, what is the spin of a single-photon state? The result you get depends on the polarization basis you choose for the mode expansion.

Problem 6.3 – Fock space, bosons, fermions, and temperature (7 points)

(i) Let $\{n\} = \{n_1, \ldots n_k, \ldots\}$ stand for the set of occupation numbers in all (photon) field modes. Show that for a stationary bosonic state, one has

$$\langle \{n\} | a_k^\dagger a_l | \{n\} \rangle = \delta_{kl} n_k, \qquad \langle \{n\} | a_k a_l^\dagger | \{n\} \rangle = \delta_{kl}(n_k + 1). \tag{6.6}$$

where n_k is the photon occuation number in mode k. Show furthermore that all other products of one or two mode operators have zero expectation value.

(ii) Find out how this result is generalized to a state (actually a density operator) in thermal equilibrium by postulating that the state $|\{n\}\rangle$ occurs with a probability proportional to $p(\{n\}) = \exp(-\beta \sum_k \omega_k n_k)$ (Boltzmann-weighted average, $T = \hbar/(k_B \beta)$). Conclude that the average photon number is given by the Bose-Einstein statistics (Fermi-Dirac statistics) if occupation numbers $n_k = 0, 1, 2, \ldots$ ($n_k = 0, 1$) are allowed for, respectively.

(iii) Calculate the average free energy density, the average photon number, the entropy, and the average number of excited modes for blackbody radiation (i.e., the electromagnetic field in thermal equilibrium).

Note. A field mode is not in its vacuum state with probability $\mathrm{e}^{-\beta\omega_k}$. Why?

Problem 6.4 – Angular average (3 points)

Calculate the polarization sum μ and the integral over the angles of the wavevector $\mathbf{k}$

$$\int \frac{\mathrm{d}\Omega(\mathbf{k})}{4\pi} \sum_\mu e_{\mathbf{k}\mu,i} e^*_{\mathbf{k}\mu,j} = \frac{2}{3}\delta_{ij} \tag{6.7}$$

that appears in the rate of spontaneous emission.

[5] e.g., W. Greiner, J. Reinhardt, *Theoretische Physik Band 7a – Feldquantisierung* or Dr. Blümlein's Quantum Field Theory lecture.