

# **Einführung in die Quantenoptik I**

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The preliminary programme for this lecture:

### I Interaction between light and atoms (matter)

- minimal coupling, electric dipole coupling
- the model of a two-level atom
- Bloch equations
- quantum states of one and two qubits, entanglement

### II QED – field quantization

- canonical scheme in the Coulomb gauge
- quantum (or vacuum) fluctuations, Casimir energy
- states of the radiation field: Fock, coherent, thermal, squeezed; distribution functions in phase space
- state transformation at a beamsplitter

### III Applications

- cavity QED: collapse and revival, quantum computing (gates)
- spontaneous emission
- photodetection

### **Outlook:** quantum optics II

- master equations, open systems, system + bath paradigm
- quantum theory of the laser
- correlations and fluctuations, spectral characterization

# Chapter 1

## Atom-light interaction

In this chapter, we review some basic physics about atoms and their interaction with light. The focus is on developing simple approximations that describe the coupling to near-resonant light fields.

To begin with, we describe in this chapter the field classically. The argument can be done as well, however, with the quantized field. We shall give here and there the corresponding formulas without going into the details.

### 1.1 Hamiltonian

An atom can be modelled as a collection of charged point particles. The simplest Hamiltonian one can write is therefore

$$H_A = \sum_{\alpha} \frac{\mathbf{p}_{\alpha}^2}{2M_{\alpha}} + \frac{1}{2} \sum_{\alpha \neq \beta} \frac{e_{\alpha} e_{\beta}}{4\pi\epsilon_0 |\mathbf{x}_{\alpha} - \mathbf{x}_{\beta}|} \quad (1.1)$$

where  $\alpha$  labels the particles,  $M_{\alpha}, e_{\alpha}$  are their masses and charges. We try to use in this lecture SI units. (In cgs units, drop the  $4\pi\epsilon_0$ .) The interaction term corresponds to the electrostatic (or Coulomb) field created by the charges.

More advanced atomic models adopt a relativistic viewpoint, take into account the electron spin, the magnetic field created by the motion of the particles, the corresponding spin-orbit interaction, the spin-spin coupling, the hyperfine interaction etc. Theoretical atomic physics computes all these corrections to the energy levels and matrix elements ‘from first principles’ (see the textbook by Haken & Wolf (2000), for example). Typically, no simple analytical results can be found for atoms with more than two or three electrons, say.

hydrogen		H	$n$	$l = 0$	$l = 1$	$l = 2$	$\dots$
lithium		Li	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$
sodium	Natrium	Na	3	3s	3p	3d	
potassium	Kalium	K	2	2s	2p		
rubidium		Rb	1	1s			
cesium		Cs					
francium		Fr					

Table 1.1: Left: the series of alkaline atoms. Right: Spectroscopic notation for energy levels of hydrogen-like atoms.

For our purposes, we are not interested in so much detail. Instead, we use a simplified description of the atom states that captures their essential properties. Good examples are atoms with a single electron in the outer shell (the alkaline series), as listed in table 1.1. Their energy levels are to a good approximation given by a modified Balmer formula

$$E_{nl} = -\frac{e^2}{8\pi\epsilon_0 a_0 (n + \delta_l)^2} = -\frac{\text{Ryd}}{(n + \delta_l)^2}, \quad \begin{array}{l} n = 1, 2, \dots \\ l = 0, \dots, n-1 \end{array} \quad (1.2)$$

The Bohr radius  $a_0 = 4\pi\epsilon_0\hbar^2/me^2 \approx 0.5 \text{ \AA}$  gives the typical size of the electron cloud, and the ‘quantum defect’  $\delta_l$  lifts the degeneracy of the hydrogen levels.<sup>1</sup> The energy scale is given by the Rydberg constant  $\text{Ryd} \approx 13.6 \text{ eV}$ .

The frequency of electromagnetic radiation emitted by atoms in a process  $|i\rangle \rightarrow |f\rangle$  is given according to Bohr by

$$\omega_{if} = \frac{E_i - E_f}{\hbar} \quad (1.3)$$

For two typical energy levels  $E_{i,f}$ ,  $\hbar\omega_{if}$  is also of the order of 1 Ryd. If we compute the wavelength of the corresponding electromagnetic radiation, we find

$$\lambda_{if} \sim \frac{2\pi\hbar c}{\text{Ryd}} = \frac{4\pi}{\alpha_{\text{fs}}} a_0 \gg a_0 \quad (1.4)$$

$$\text{fine structure constant } \alpha_{\text{fs}} = \frac{e^2}{4\pi\epsilon_0\hbar c} \approx \frac{1}{137}, \quad (1.5)$$

<sup>1</sup>The charge  $Z|e|$  of the nucleus enters this formula via the quantum defects. In fact, the outer electron ‘sees’ the nucleus screened by the core electrons. This gives a Coulomb potential as for the hydrogen atom, with some modifications due to the core electrons. These are responsible for the lifted degeneracy between the  $l$  states.

which is much longer than the typical size of the atom (the Bohr radius gives the extension of the electronic orbitals). For typical light fields, the atom thus appears like a pointlike object. This property justifies the ‘long wavelength approximation’ that simplifies the Hamiltonian for the atom–light interaction.

Another way of looking at the result (1.4) is to interpret the inverse fine structure constant,  $\alpha_{\text{fs}}^{-1} = c(4\pi\epsilon_0\hbar/e^2)$  as the ratio between the speed of light  $c$  and the typical velocity for an electron in the Hydrogen atom (the natural velocity scale in the so-called ‘atomic units’). We see that this velocity is only a few percent of  $c$ , hence we expect that the non-relativistic description we have used so far is a good approximation.

## 1.2 Atom-light interaction

### 1.2.1 Minimal coupling

According to the rules of electrodynamics, the interaction between a collection of charges with a given electromagnetic field is described by the ‘minimal coupling’ Hamiltonian. This corresponds to the replacement  $\mathbf{p}_\alpha \mapsto \mathbf{p}_\alpha - e_\alpha \mathbf{A}(\mathbf{x}_\alpha, t)$  where  $\mathbf{A}(\mathbf{r}, t)$  is the vector potential. In this chapter, this is a given time-dependent function. It will become an operator when the field is quantized. In addition, there is the potential energy due to an ‘external’ scalar potential  $\phi_{\text{ext}}(\mathbf{x}, t)$ , so that we get

$$H_{AF} = \sum_{\alpha} \frac{(\mathbf{p}_\alpha - e_\alpha \mathbf{A}(\mathbf{x}_\alpha, t))^2}{2M_\alpha} + \frac{1}{2} \sum_{\alpha \neq \beta} \frac{e_\alpha e_\beta}{4\pi\epsilon_0 |\mathbf{x}_\alpha - \mathbf{x}_\beta|} + \sum_{\alpha} e_\alpha \phi_{\text{ext}}(\mathbf{x}_\alpha, t). \quad (1.6)$$

The minimal coupling prescription is related to the freedom of choosing the phase reference of the wave function, as is seen in more detail in the exercises.

**Remark.** This freedom is also called ‘local  $U(1)$  gauge invariance’ because phase factors form the unitary group  $U(1)$ . Local changes in the phase of the wave function generate terms in the Schrödinger equation that can be combined with gauge transformations for the electromagnetic potentials. This connection to the electromagnetic gauge transformations is of great importance for quantum field theory. It allows to construct the coupling to the electromagnetic field from the symmetry properties of the quantum fields. For example, there are theories where electrons and neutrinos are combined into a two-component field, and the interactions are invariant under  $SU(2)$  transformations that mix these two, plus  $U(1)$  transformations of the phase common to the two components. The group  $SU(2) \times U(1)$  is four-dimensional and has four ‘generators’. Each of them corresponds to a vector potential that interacts with the two-component field. In addition

to the standard electromagnetic potential (the ‘photon’), there are interactions associated to the ‘massive vector bosons’, called  $W^\pm$  and  $Z^0$ . They convey the ‘weak interaction’ that is responsible for  $\beta$  decay. More details in any book on quantum field theory. I sometimes use the one by Itzykson & Zuber (2006).

When the minimal coupling Hamiltonian (1.6) is expanded to lowest order in the charges  $e_\alpha$ , we obtain the so called ‘ $p \cdot A$ ’ interaction

$$H_{\text{int}} = - \sum_{\alpha} \frac{e_{\alpha}}{2M_{\alpha}} \{ \mathbf{p}_{\alpha} \cdot \mathbf{A}(\mathbf{x}_{\alpha}, t) + \mathbf{A}(\mathbf{x}_{\alpha}, t) \cdot \mathbf{p}_{\alpha} \}. \quad (1.7)$$

In the Coulomb gauge where  $\nabla \cdot \mathbf{A} = 0$ , the ordering of the operators is irrelevant. This interaction is linear in the vector potential, but there is also a second-order (or ‘diamagnetic’) term

$$H_{\text{dia}} = \sum_{\alpha} \frac{e_{\alpha}^2 \mathbf{A}^2(\mathbf{x}_{\alpha}, t)}{2M_{\alpha}} \quad (1.8)$$

When calculations are pushed to second order in the  $p \cdot A$ -coupling, the diamagnetic interaction must be included as well, for consistency. This makes the ‘book-keeping’ in perturbation theory complicated.

### 1.2.2 Electric dipole coupling

A simpler approach is possible, however. We first make the approximation that the field varies slowly on the scale of the displacements  $\mathbf{x}_{\alpha}$  of the charges in the atom. Then we can replace, to lowest order,  $\mathbf{A}(\mathbf{x}_{\alpha}, t) \approx \mathbf{A}(\mathbf{R}, t)$  where  $\mathbf{R}$  is the atomic center of charge. This is called the ‘long-wavelength approximation’ that is well justified for fields near-resonant with typical atomic transitions. Within this approximation, we can find a simpler interaction Hamiltonian that is linear in the electromagnetic field. It is called the ‘ $d \cdot E$ ’ coupling and is strictly linear in the electric field.

$$H_{\text{int}} = -\mathbf{d} \cdot \mathbf{E}(\mathbf{R}, t), \quad (1.9)$$

$$\mathbf{d} = \sum_{\alpha} e_{\alpha}(\mathbf{x}_{\alpha} - \mathbf{R}), \quad (1.10)$$

where  $\mathbf{d}$  is the electric dipole moment of the atom relative to  $\mathbf{R}$ . This version of the  $d \cdot E$  interaction can be derived from the minimal coupling Hamiltonian with a gauge transformation (see the exercises) from the minimal coupling Hamiltonian in the long-wavelength approximation, without invoking an additional approximation.

The advantages of the electric dipole coupling are: the atom couples directly to the field; there is no quadratic interaction term. One must not forget that between the two interactions, the wave function (the atomic state) differs by a unitary transformation. Otherwise, some matrix elements or transition rates may come out differently. This issue is discussed in great detail in the book ‘Molecular Quantum Electrodynamics’ by Craig & Thirunamachandran (1984) and in Chap. IV of ‘Photons and Atoms – Introduction to Quantum Electrodynamics’ by Cohen-Tannoudji & al. (1987).

### 1.2.3 Selection rules

Since the electric dipole moment determines the interaction with the light field, a few remarks on its matrix elements are in order. We take as a starting point the basis of the stationary states of an atom, described by the Hamiltonian (1.1). These states are typically described by quantum numbers like parity, angular momentum etc. The ‘selection rules’ specify for which states we know by symmetry that the matrix elements of the electric dipole moment vanish. In that case, the corresponding states are not connected by an ‘electric dipole transition’, or the transition is ‘dipole-forbidden’.

**Parity.** If the state  $|a\rangle$  has a defined parity  $P_a = \pm 1$  (i.e., the electronic wave function  $\psi(\{\mathbf{x}_\alpha\})$  transforms like  $P_a \psi(\{\mathbf{x}_\alpha\})$  when all coordinates are transformed as  $\mathbf{x}_\alpha \mapsto -\mathbf{x}_\alpha$ ), then  $\langle a|\mathbf{d}|a\rangle = 0$ . In addition, the matrix element  $\langle a|\mathbf{d}|b\rangle$  is only nonzero when  $|a\rangle$  and  $|b\rangle$  have different parity.

*Angular momentum.* If there is no electron spin, this is given by  $l$ , and by  $j = l \pm \frac{1}{2}$  for hydrogen-like atoms where one spin of a non-paired electron is present. The vector operator  $\mathbf{d}$  transforms under rotation like a spin 1 (there are three different basis vectors). One can introduce a basis  $\mathbf{e}_q$  ( $q = -1, 0, 1$ ) that are eigenvectors of  $L_3$  as well and write  $\mathbf{d} = \sum_q d_q \mathbf{e}_q$ . The product  $\mathbf{e}_q|l, m\rangle$  then is an eigenstate of  $L_3$  with eigenvalue  $q + m$ . Therefore, the matrix element with  $|l', m'\rangle$  is only nonzero when  $m' = q + m$ . We find the selection rule

$$|m - m'| \leq 1.$$

In addition, the product states  $\mathbf{e}_q|l, m\rangle$  can be expanded onto eigenstates of  $\mathbf{L}^2$ . The rules for the ‘addition of angular momentum’ imply that only angular momenta  $l' = l - 1, l, l + 1$  occur in this expansion. This gives the selection rule

$$|l - l'| \leq 1.$$

*Total momentum.* An atom that is in a plane wave state regarding its centre-of-mass motion, with momentum  $\mathbf{P}$  receives an additional momentum  $\hbar\mathbf{k}$  when a photon from a plane electromagnetic wave with wave vector  $\mathbf{k}$  is absorbed. The corresponding ‘recoil velocity’  $\hbar\mathbf{k}/M$  is of the order of a few mm/s to a few cm/s for typical atoms. The atomic recoil plays an important role for atom deceleration and cooling with laser light.

## 1.3 Two-level atoms

For the rest of this lecture, it will be sufficient to write the atomic Hamiltonian in the form

$$H_A = \sum_n E_n |n\rangle\langle n|, \quad (1.11)$$

where the states  $|n\rangle$  are the stationary states corresponding to the energy eigenvalue  $E_n$ . But even this form is too complicated: it contains too many terms when dealing with near-resonant laser light. This is the setting we shall focus on here. One can then retain only a few states to describe the atom.

### 1.3.1 Two-level language

#### Two-level observables

**Atomic Hamiltonian.** The standard notation for the two states is  $|g\rangle$  for the ground state and  $|e\rangle$  for the excited state. The Bohr frequency is often written  $\omega_A = \omega_e - \omega_g > 0$ . The atomic energy levels are often referenced to a zero energy lying between both states, this gives:

$$H_A = \frac{\hbar\omega_A}{2} |e\rangle\langle e| - \frac{\hbar\omega_A}{2} |g\rangle\langle g| \quad (1.12)$$

It is also useful to identify the two-dimensional Hilbert space of the two-level atom with the  $\mathbb{C}^2$ , using the basis vectors  $(1, 0)^T \leftrightarrow |e\rangle$  and  $(0, 1)^T \leftrightarrow |g\rangle$ . The Hamiltonian then becomes the diagonal matrix

$$H_A = \frac{\hbar\omega_A}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \frac{\hbar\omega_A}{2} \sigma_3 \quad (1.13)$$

where  $\sigma_3$  is the third Pauli matrix. Indeed, it is obvious that a two-dimensional Hilbert space can be identified with the Hilbert space of a spin 1/2.

Observables energy, inversion, dipole.

$$\mathbf{d} = \mathbf{d}_{ge}\sigma + \mathbf{d}_{ge}^*\sigma^\dagger = \mathbf{d}_{ge}|g\rangle\langle e| + \mathbf{d}_{ge}^*|e\rangle\langle g| \quad (1.14)$$

where the vector of matrix elements of the dipole operator is  $\mathbf{d}_{ge} = \langle g|\mathbf{d}|e\rangle$ . Only off-diagonal matrix elements because of the parity selection rule.

### State of a two-level system

The expectation values  $\langle\sigma_3\rangle$  and  $\langle\sigma\rangle$  completely specify the state of the two-level system.

Why is this so? A general observable is a hermitean  $2 \times 2$  matrix. All these matrices are linear combinations of Pauli matrices

$$\begin{aligned} A &= \begin{pmatrix} a_{ee} & a_{eg} \\ a_{ge} & a_{gg} \end{pmatrix} = \frac{a_{ee} + a_{gg}}{2} \mathbb{1} + \frac{a_{ee} - a_{gg}}{2} \sigma_3 + a_{ge} \sigma + a_{eg} \sigma^\dagger \\ &= \frac{a_{ee} + a_{gg}}{2} \mathbb{1} + \sum_j a_j \sigma_j, \end{aligned} \quad (1.15)$$

$$\sigma_1 = \sigma + \sigma^\dagger \quad (1.16)$$

$$\sigma_2 = i(\sigma - \sigma^\dagger) \quad (1.17)$$

with real coefficients  $a_j$ .

The above statement is true even for a more general definition of a *state* than you may be used to. In the axiomatic language of quantum information, a state is a mapping from a set of observables to their expectation values

$$\rho : A \mapsto \rho(A) = \langle A \rangle_\rho \quad (1.18)$$

Linear map with  $\rho(\mathbb{1}) = 1$  (real or complex coefficients depending on choice of observable algebra) and  $\rho(A)$  real for a hermitean  $A$ .

Now, the action of this map is determined by evaluation on basis vectors = Pauli matrices for a two-level system:

$$\rho(A) = \frac{a_{ee} + a_{gg}}{2} \rho(\mathbb{1}) + \sum_j a_j \rho(\sigma_j) = \frac{a_{ee} + a_{gg}}{2} \rho(\mathbb{1}) + \sum_j a_j s_j \quad (1.19)$$

with components of *Bloch vector*  $\mathbf{s} = (s_1, s_2, s_3)$ .

This definition is more general than complex linear combinations of  $|e\rangle$  and  $|g\rangle$ . These states play a special role and are called *pure states*. They also correspond to special observables: *projectors*

$$\mathbb{P}_\phi = |\phi\rangle\langle\phi| \quad (1.20)$$

This is also a hermitean operator with eigenvalues 0 or 1. A physical state has the property

$$\rho(\mathbb{P}_\phi) \geq 0 \quad \text{for all } |\phi\rangle \quad (1.21)$$

Physical interpretation: this is the probability of finding the system in the pure state  $|\phi\rangle$ , which clearly must be a positive number.

Definition of *density matrix* (or *density operator*): any linear map on the vector space of observables can be represented by a suitable linear form

$$\rho(A) = \text{tr}(\bar{\rho}A) \quad (1.22)$$

where  $\bar{\rho}$  is a hermitean operator. This rule corresponds to the usual calculation of expectation values for mixed states in quantum statistics. In a finite-dimensional system, it corresponds to the duality between linear forms and vectors: each linear form can be represented as a scalar product with a suitable vector. This becomes the Riesz representation theorem in an infinite-dimensional Hilbert space.

Using this for projector observables, we find from Eq.(1.21):

$$0 \leq \rho(\mathbb{P}_\phi) = \text{tr}(\bar{\rho}|\phi\rangle\langle\phi|) = \langle\phi|\bar{\rho}|\phi\rangle \quad (1.23)$$

Hence the diagonal elements of the density matrix are positive, in any basis. This connects again to the interpretation of the probability of finding the system in the state  $|\phi\rangle$ .

Density operator as observable itself. Expectation value is called purity

$$\text{Pu}(\rho) = \langle\bar{\rho}\rangle_\rho = \text{tr}(\bar{\rho}^2) = \dots = \frac{1}{2}(1 + \mathbf{s}^2) \quad (1.24)$$

Calculation uses representation in terms of Bloch vector and Pauli matrices

$$\bar{\rho} = \frac{1}{2} \left( \mathbb{1} + \sum_j s_j \sigma_j \right) = \frac{\mathbb{1} + \mathbf{s} \cdot \boldsymbol{\sigma}}{2} \quad (1.25)$$

## Bloch sphere

The language of a spin 1/2 is often used to visualize the dynamics of a two-level atom. Let us compute the components of the average spin vector  $(s_1, s_2, s_3)^T \equiv \langle\boldsymbol{\sigma}\rangle$  (the ‘Bloch vector’) in the pure state  $|\psi(t)\rangle = (c_e(t), c_g(t))^T$ :

$$s_3 = |c_e|^2 - |c_g|^2 \quad (1.26)$$

$$s_1 - is_2 = 2c_g^* c_e \quad (1.27)$$

Observe that  $s^2 = s_1^2 + s_2^2 + s_3^2 = 1$  for a pure state. The component  $s_3$  is related to the occupation probabilities  $|c_e(t)|^2$ ,  $|c_g(t)|^2$  (the ‘populations’): it gives the *inversion*, i.e., the difference of the ground and excited state populations. In the ground state, one has  $\langle \sigma_3 \rangle = -1$ . The other two components (1.27) are only nonzero when the atom is in a superposition of the ground and excited states.

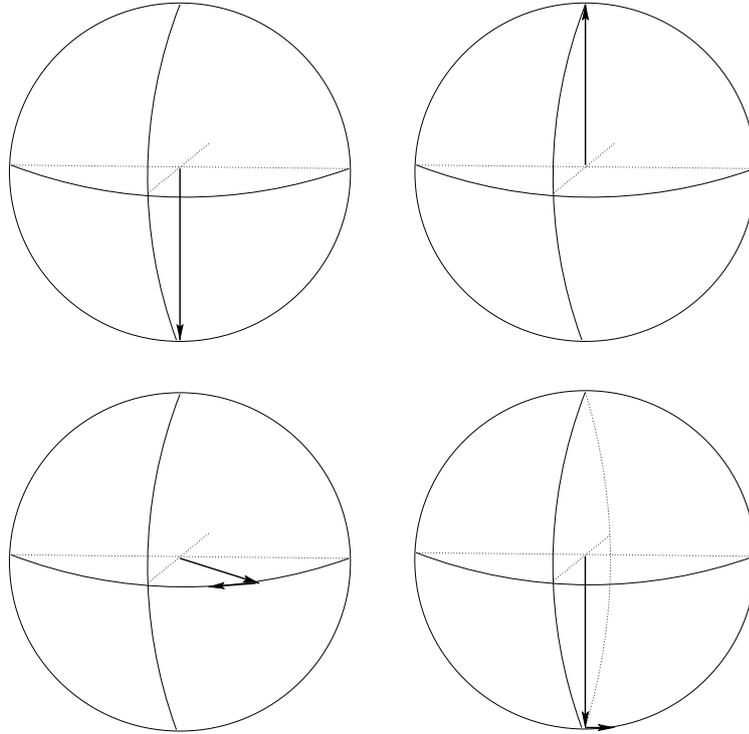


Figure 1.1: Upper left: Bloch vector for an atom in the ground state. Upper right: excited state. Lower left: superposition of ground and excited states with equal weight and free rotation without a laser field. Lower right: start of resonant Rabi oscillations with an initial ground state. The  $x_1$ -axis points to the right, the  $x_3$ -axis upwards.

In the exercises, you show that the Bloch vector is in general constrained by  $s^2 \leq 1$ , due to the positivity condition (1.23) of the density matrix. It can thus be represented by a point in a sphere, the ‘Bloch sphere’ (see Fig.1.1). On the sphere surface are located the pure states: the ‘south pole’ corresponds to an atom in the ground state, while an excited atom is at the ‘north pole’. In between these two, the atom is in a superposition of ground and excited states. In particular, around the equator, both ground and excited state are occupied with probability

1/2 (the inversion is zero). The longitude along the equator is fixed by the phase difference between ground and excited state.

If no external laser field is applied, the Bloch vector rotates at the frequency  $\omega_A$  clockwise around the vertical axis. In particular, the occupation probabilities do not change with time: the inversion, being the projection onto the vertical axis, is constant. With the laser on, we shall see that the rotation axis gets tilted so that an atom initially in the ground state develops an excited component.

### 1.3.2 Resonance approximation

#### Interaction Hamiltonian

The interaction with a monochromatic laser field can be described by the Hamiltonian

$$\begin{aligned} H &= H_A - \mathbf{d} \cdot \mathbf{E}(t) \\ \mathbf{E}(t) &= \mathbf{E} e^{-i\omega_L t} + \text{c.c.} \\ \mathbf{d} &= \mathbf{d}_{ge} |g\rangle\langle e| + \text{h.c.} \end{aligned} \quad (1.28)$$

where the complex vector  $\mathbf{E}$  gives the amplitude of the electric field. It is evaluated at the position of the atom, we drop this dependence here. The laser (angular) frequency is  $\omega_L$ . The term  $\mathbf{E} e^{-i\omega_L t}$  is called the ‘positive frequency part’ of the field: its time evolution is the same as for a solution of the time-dependent Schrödinger equation (with positive energy  $\hbar\omega_L$ ).

We re-write Eq.(1.28) in terms of two separately hermitean operators

$$\begin{aligned} -\mathbf{d} \cdot \mathbf{E}(t) &= \frac{\hbar}{2} \left( \Omega e^{-i\omega_L t} |e\rangle\langle g| + \text{h.c.} \right) \\ &\quad + \frac{\hbar}{2} \left( \Omega' e^{i\omega_L t} |e\rangle\langle g| + \text{h.c.} \right) \end{aligned} \quad (1.29)$$

$$\frac{\hbar\Omega}{2} = -\mathbf{d}_{ge}^* \cdot \mathbf{E}, \quad (\text{Rabi frequency}) \quad (1.30)$$

where  $\Omega$  is a complex-valued frequency, and  $\Omega'$  has a similar expression as Eq.(1.30). (We follow the Paris convention and notation for  $\Omega$ .)

If the field is quantized, Eq.(1.28) applies in similar form in the interaction picture and involves photon annihilation operators  $a_L$  in place of the complex amplitude  $\mathbf{E}$  (and creation operators  $a_L^\dagger$  in place of  $\mathbf{E}^*$ ). The fully quantized interaction Hamiltonian is derived from the quantized

field operator and takes the form:

$$-\mathbf{d}(t) \cdot \mathbf{E}(t) \mapsto -\sum_k \mathcal{E}_k \left( a_k(t) \mathbf{f}_k \cdot \mathbf{d}(t) + a_k^\dagger(t) \mathbf{f}_k^* \cdot \mathbf{d}(t) \right) \quad (1.31)$$

where  $\mathcal{E}_k = \sqrt{\hbar\omega_k/(2\varepsilon_0V)}$  is the electric field amplitude at the one-photon level and  $\mathbf{f}_k = \mathbf{f}_k(\mathbf{R})$  is the normalized mode function, evaluated at the position  $\mathbf{R}$  of the atom. We have written Eq.(1.31) in the “interaction picture” where all operators carry their “free” time dependence. For the photon annihilation operator  $a_k(t) = a_k e^{-i\omega_k t}$ , which is the operator in the Heisenberg picture under the free field Hamiltonian. The time dependence of the (freely evolving) dipole operator is given by  $\mathbf{d}(t) = \exp(iH_A t) \mathbf{d} \exp(-iH_A t)$ .

In order to examine what happens to an atom illuminated by a laser field, we make the *Ansatz*

$$|\psi(t)\rangle = \tilde{c}_e(t) e^{-i\omega t/2} |e\rangle + \tilde{c}_g(t) e^{+i\omega t/2} |g\rangle \quad (1.32)$$

where the frequency  $\omega$  is chosen later. The amplitudes describe the two-level system in a *picture* that differs from the usual choice  $c_e(t), c_g(t)$ . The Schrödinger equation for  $\tilde{c}_e$  contains a correction term because of the time-dependent exponential. One gets

$$i\hbar\partial_t \tilde{c}_e = \left( E_e - \frac{\hbar\omega}{2} \right) \tilde{c}_e + \frac{\hbar}{2} \Omega e^{-i(\omega_L - \omega)t} \tilde{c}_g + \frac{\hbar}{2} \Omega' e^{i(\omega_L + \omega)t} \tilde{c}_g \quad (1.33)$$

Now, we can take the choice  $E_e = \frac{1}{2}\hbar\omega_A$  for the excited state energy. There are now two “natural choices” for  $\omega$ :

- (1) **interaction picture:**  $\omega = \omega_A$ , and the first term disappears. The time dependence of  $\tilde{c}_e$  is then only due to the atom-laser interaction. This picture is suitable for perturbation theory.
- (2) **rotating frame:**  $\omega = \omega_L$ , and the second term becomes time-independent. This picture is suitable for concrete calculations in quantum optics, once we have convinced ourselves what is the meaning of the third term.

### Time-dependent perturbation theory

We now make the choice (1) of the interaction picture and solve the equation for  $\tilde{c}_e$  with the help of time-dependent perturbation theory. This proceeds by identifying the interaction Hamiltonian as a “small term” and by counting ascending powers.

At the order zero,  $H_A$  is the only Hamiltonian. Keeping in mind the choice  $\omega = \omega_A$ , Eq.(1.33) reduces to

$$\partial_t \tilde{c}_e^{(0)} = 0, \quad \partial_t \tilde{c}_g^{(0)} = 0 \quad (1.34)$$

where the equation for  $\tilde{c}_g$  is similar to Eq.(1.33). As expected from the analogy to the time-dependent Schrödinger equation (for the “free” atom), the amplitudes are constants at order zero. The natural initial condition “atom is in state  $|g\rangle$ ” translates into

$$\tilde{c}_e^{(0)}(t) = 0, \quad \tilde{c}_g^{(0)}(t) = 1. \quad (1.35)$$

To the first order, we get from Eq.(1.33)

$$i\hbar \partial_t \tilde{c}_e^{(1)} = \frac{\hbar}{2} \Omega e^{-i(\omega_L - \omega_A)t} \tilde{c}_g^{(0)} + \frac{\hbar}{2} \Omega' e^{i(\omega_L + \omega_A)t} \tilde{c}_g^{(0)} \quad (1.36)$$

Now, since we know  $c_g^{(0)}(t)$  as a (constant) function of time, this can be integrated immediately to give

$$c_e(t) = c_e(0) + \frac{\Omega}{2} \frac{e^{-i(\omega_L - \omega_A)t} - 1}{\omega_L - \omega_A} - \frac{\Omega'}{2} \frac{e^{i(\omega_A + \omega_L)t} - 1}{\omega_A + \omega_L} \quad (1.37)$$

The two terms in this result have distinct physical interpretations, related to the denominators.

**Absorption.** The first denominator leads to a ‘large’ result when  $\omega_A = \omega_L$ . One says that the atom went from the state  $|g\rangle$  to the higher-lying state  $|e\rangle$  by absorbing one ‘energy quantum’ (‘photon’). (Recall that the amplitude  $c_e$  for the state  $|e\rangle$  is increased in Eq.(??).) This process is governed by the ‘positive frequency’ component  $\Omega e^{-i\omega_L t}$  of the interaction Hamiltonian (corresponding to the positive frequency component of the electromagnetic field). In the quantized description of the light field, this component corresponds to an ‘annihilation operator’ that removes one photon from the field. If we fix the states  $|g\rangle$  and  $|e\rangle$  such that the condition for absorption is satisfied, then the second term in Eq.(1.37) has a ‘large’ denominator,  $\omega_A + \omega_L \approx 2\omega_A$ . This term is therefore much smaller than the first one, by a factor of the order  $\mathcal{O}(10^{-6})$  for laser fields of reasonable intensity (see exercise). This suggests that we can neglect this term. This approximation is called the ‘resonance approximation’ (or the ‘rotating wave approximation’, an admittedly strange name). If we keep the non-resonant term, we deal in the quantum theory with a ‘virtual’ process where the atom passes into a state with a higher energy and at the same time, a photon is created.

**Emission.** If we had started with the atom in the excited state  $|e\rangle$ , one would get a resonant contribution again for  $\omega_L = \omega_A$ , with a large amplitude being created in  $|g\rangle$ . This corresponds to a transition with the energy balance  $E_e = E_g + \hbar\omega_L$ : the atom makes a transition to a lower-lying state, and in the quantized field description, a ‘photon’ is created (by the creation operator  $a_k^\dagger$  in the expansion of the field operator). The non-resonant term in this setting would correspond to the atom decaying to the ground state and absorbing a photon, clearly a virtual process.

To summarize, in the resonance approximation, we only retain those parts of the interaction Hamiltonian where the excitation of the atom (the operator  $\sigma^\dagger = |e\rangle\langle g|$ ) is accompanied by a positive frequency laser field  $\mathbf{E}e^{-i\omega_L t}$ . This approximation is consistent with the two-level approximation where right from start, we only considered atomic levels whose Bohr frequencies are near-resonant with the laser.

This approximation is possible for a ‘detuning’  $\Delta = \omega_L - \omega_A$  small compared to the typical differences between atomic transition frequencies. This condition is easily achieved since transition frequencies (spectral lines) differ easily by energies of the order of 1 eV, and this is a ‘huge’ detuning to drive an atomic transition.

**Remark.** The description of absorption and emission, as we encounter it here, does not explicitly require the quantization of the light field. These processes also occur in a ‘classical’ time-dependent potential because energy is not conserved there, as is well known in classical mechanics. One can push the analogy even further: a weak monochromatic excitation of a mechanical system reveals the system’s ‘resonance frequencies’. For an atom, these are apparently given by the Bohr frequencies. The only difference to a mechanical system is that we are inclined to use different names for the excitations with positive and negative frequencies, since in the atomic energy spectrum, there is a definite difference between ‘going up’ and ‘going down’ (there exists a ground state).

### Atomic polarizability

The perturbative calculation can be used to determine the dipole moment that the laser field induces in the atom. This dipole is, within the lowest order in the perturbation, linear in the amplitude  $\mathbf{E}$  of the laser, or equivalently, in the Rabi frequency  $\Omega$ . As is discussed in more detail in the exercises (Problem 3.2), the polarizability is defined by equating the average (induced) dipole moment with

a function linear in the laser amplitude,

$$\langle \psi(t) | (\mathbf{d}_{ge}\sigma + \text{h.c.}) | \psi(t) \rangle = \alpha(\omega_L) \mathbf{E} e^{-i\omega_L t} + \text{h.c.} \quad (1.38)$$

Note that the asymptotic regime  $t \rightarrow \infty$  is taken here where the atomic dipole oscillates at the frequency of the external field. Now, for the initial condition that the atom starts in its ground state, the polarizability is

$$\alpha_g(\omega) = \frac{(2\omega_{eg}/\hbar) \mathbf{d}_{ge} \otimes \mathbf{d}_{ge}^*}{\omega_{eg}^2 - \omega^2} \quad (1.39)$$

where two peaks at  $\omega = \pm\omega_{eg}$  appear.

### RWA Hamiltonian in the rotating frame

We come back to the (near-)resonant interaction: it can be described by the (effective) Hamiltonian

$$H_{AL} = -\mathbf{d}_{ge}^* \mathbf{E} e^{-i\omega_L t} \sigma^\dagger - \mathbf{d}_{ge} \mathbf{E}^* e^{i\omega_L t} \sigma \quad (1.40)$$

This is called the ‘‘rotating wave approximation’’, a physically more transparent name would be ‘‘resonance approximation’’. The change into the picture (2) mentioned above ( $\omega = \omega_L$ ) corresponds to the unitary transformation

$$|\psi(t)\rangle = e^{-i\frac{\omega_L t}{2} \sigma_3} |\tilde{\psi}(t)\rangle \quad (1.41)$$

This gives for the state  $|\tilde{\psi}(t)\rangle$  the following Hamiltonian

$$H = -\frac{\hbar\Delta}{2} \sigma_3 + \frac{\hbar}{2} \Omega \sigma^\dagger + \frac{\hbar}{2} \Omega^* \sigma, \quad \Delta = \omega_L - \omega_A \quad (1.42)$$

where the time-dependence of Eq.(1.40) has disappeared (exactly) and where only the *detuning*  $\Delta$  instead of the laser frequency appears (Paris convention for the sign of  $\Delta$ ).

With a quantized field, add the Hamiltonian  $H_F$  for the field and make the replacement

$$\frac{\hbar}{2} (\Omega^* \sigma + \sigma^\dagger \Omega) \mapsto - \sum_k \mathcal{E}_k \left\{ a_k^\dagger \mathbf{f}_k^* \cdot \mathbf{d}_{eg}^* \sigma + \sigma^\dagger a_k \mathbf{f}_k \cdot \mathbf{d}_{eg} \right\} \quad (1.43)$$

One could also read this as an operator-valued Rabi frequency per mode,  $\hat{\Omega}_k$ .

### 1.3.3 Overview

The Hamiltonian (1.42), combined with its quantized field formulation, is the basis for most of the phenomena that have been studied in the quantum optics of two-level system. One can discuss the following topics (we give a selection in this lecture):

- Rabi oscillations in a classical monochromatic field;
- spontaneous decay of an excited atom into the continuum of vacuum field modes (initially in the ground state);
- collapse and revival of Rabi oscillations when the atom couples to a single quantized field mode. The collapse and the revival occurs because the Rabi frequency depends on the photon number, and the oscillations for the different Fock state components of a field state (a coherent state, for example) get out of phase;
- resonance fluorescence is the radiation emitted by an atom driven by a near-resonant laser field. This combines Rabi oscillations in a classical field with the emission of photons into the empty mode continuum. Of particular interest is the spectrum of this emission: it contains, for sufficiently strong driving, two sidebands, split by the Rabi frequency from the central line (centered at the laser frequency). The central line contains a monochromatic component ('elastic scattering', related to the laser-induced dipole moment as in classical electrodynamics) and a broadened component of Lorentzian shape, related to spontaneous emission. This spectrum is a cornerstone of quantum optics and one of the few examples of a non-perturbative calculation in quantum electrodynamics.

## 1.4 Spin 1/2 analogy

We now come back to the spin 1/2 analogy. The Hamiltonian (1.40) with the atomic energies and the atom-laser interaction has the same form as the Hamiltonian for a spin 1/2 in a time-dependent magnetic field,

$$H_{\text{spin}} = \boldsymbol{\sigma} \cdot \mathbf{B}(t) \quad (1.44)$$

where  $\boldsymbol{\sigma}$  is the vector of Pauli matrices and the 'magnetic field'  $\mathbf{B}(t)$  actually has the dimensions of an energy (we took a unity magnetic moment). The magnetic

field rotates at the laser frequency around the  $x_3$ -axis:

$$\mathbf{B}(t) = \frac{\hbar}{2} \begin{pmatrix} \Omega \cos \omega_L t \\ \Omega \sin \omega_L t \\ \omega_A \end{pmatrix} \quad (1.45)$$

It is useful to change the coordinate frame such that it co-rotates with this field (this is the ‘rotating frame’). In this frame, the ‘effective magnetic field’ is static<sup>2</sup>,

$$\mathbf{B}_{\text{eff}} = \frac{\hbar}{2} \begin{pmatrix} \Omega \\ 0 \\ \omega_A \end{pmatrix}. \quad (1.46)$$

The transformation into the rotating frame also changes the wave function of our two-state particle by a unitary transformation

$$U(t) = \exp\{-i\omega_L t \sigma_3 / 2\} = \begin{pmatrix} e^{-i\omega_L t / 2} & 0 \\ 0 & e^{i\omega_L t / 2} \end{pmatrix} \quad (1.47)$$

We observe that this is the transformation we used in Eq.(1.32) to go into the interaction picture (on resonance where  $\omega_L = \omega_A$ ). This unitary transformation being time-dependent, we get also a modification of the Hamiltonian proportional to  $-i\hbar U^\dagger \partial_t U = -\hbar\omega_L \sigma_3$ . All told, we find the Hamiltonian *in the rotating frame*

$$H = -\frac{\hbar\Delta}{2}\sigma_3 + \frac{\hbar\Omega}{2}\sigma_1 \quad (1.48)$$

where the *detuning* is given by the difference between the laser frequency and the atomic transition frequency

$$\Delta = \omega_L - \omega_A. \quad (1.49)$$

Note that the laser and atomic frequencies have disappeared from the Hamiltonian and only their difference (the detuning) occurs. As a consequence, the relevant time scales (given by  $1/\Delta$  and  $1/\Omega$ ) are typically much longer than the optical period  $2\pi/\omega_L$ . On these long time scales, nonresonant processes remain ‘virtual’ and cannot be directly observed. This is consistent with the neglect of nonresonant levels (two-state approximation) and of the nonresonant two-state coupling (rotating wave approximation).

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<sup>2</sup>If we had kept the nonresonant terms in the Hamiltonian, the magnetic field would also show a time-dependent component rotating at the frequency  $2\omega_L$ .

## 1.5 Two-level dynamics

### 1.5.1 Rabi oscillations

The most simple case of atom-laser dynamics is a laser ‘on resonance’, i.e.,  $\omega_L = \omega_A$ . The Schrödinger equation for the Hamiltonian (1.42) yields (we drop the tildes)

$$i\hbar\dot{c}_e = \frac{\hbar\Omega}{2}c_g \quad (1.50)$$

$$i\hbar\dot{c}_g = \frac{\hbar\Omega}{2}c_e. \quad (1.51)$$

where the Rabi frequency is chosen real for simplicity. With the initial conditions  $c_g(0) = 1$ ,  $c_e(0) = 0$ , the solution is

$$\dot{c}_e = -i \sin(\Omega t/2) \quad (1.52)$$

$$c_g = \cos(\Omega t/2). \quad (1.53)$$

The excited state probability thus oscillates between 0 and 1 at a frequency  $\Omega/2$ . This phenomenon is called ‘Rabi flopping’. It differs from what one would guess from ordinary time-dependent perturbation theory where one typically gets linearly increasing probabilities. That framework, however, applies only if the final state of the transition lies in a continuum which is not the case here. Rabi flopping also generalizes the perturbative result (1.36) which would give a quadratic increase  $|c_e|^2 \propto t^2$  that cannot continue for long times. But instead of saturating, the atomic population returns to the ground state.

Every experimentalist is very happy when s/he observes Rabi oscillations. It means that any dissipative processes have been controlled so that they happen at a slower rate. In a realistic setting, one gets a damping of the oscillation amplitude towards equilibrium populations.

**Rabi pulses.** Rabi oscillations with a fixed interaction time are often used to implement coherent operations on an atom or spin. The corresponding evolution operator is given by (we focus on the resonant case)

$$U_\theta = \exp\{-i\theta\sigma_1/2\} = \cos(\theta/2) - i\sigma_1 \sin(\theta/2) \quad (1.54)$$

with  $\theta = \Omega t$ . After one cycle of Rabi oscillations,  $\Omega t = 2\pi$  (a ‘ $2\pi$ -pulse’), the atom returns to its ground state — but its wave function has changed sign. This

sign change is well-known from spin 1/2 particles: the corresponding unitary transformation reads

$$U_{2\pi} = \cos(\pi) - i\sigma_1 \sin(\pi) = -1 \quad (1.55)$$

A more interesting manipulation is a ‘ $\pi$ -pulse’,  $\Omega t = \pi$ , that flips the ground and excited state:

$$U_{\pi} = \cos(\pi/2) - i\sigma_1 \sin(\pi/2) = -i\sigma_1 \quad (1.56)$$

Finally, a ‘ $\pi/2$ -pulse’ takes the atom into a superposition of ground and excited states with equal weight (a Bloch vector on the equator of the Bloch sphere)

$$U_{\pi/2} = \cos(\pi/4) - i\sigma_1 \sin(\pi/4) = \frac{1 - i\sigma_1}{\sqrt{2}}$$

$$U_{\pi/2}|g\rangle = \frac{1}{\sqrt{2}}|g\rangle - \frac{i}{\sqrt{2}}|e\rangle$$

If the laser is shut off after such a pulse, the Bloch vector will continue to rotate along the equator at the frequency  $\Delta$ .

## 1.5.2 Collapse and revival

If the light field is described as a single quantized mode, an additional feature occurs in the Rabi oscillations. The key point is that the coupling Hamiltonian,  $g(a^\dagger\sigma + \sigma^\dagger a)$ , now couples the states  $|g, n\rangle$  and  $|e, n-1\rangle$  where  $n$  is the photon number. These states are split (on resonance) in energy by the ‘Rabi splitting’  $g\sqrt{n}$ . Recall that this splitting was  $|\Omega|$  for a classical laser field, proportional to the field amplitude. This is mimicked by the scaling with  $\sqrt{n}$  since the photon number  $n$  is proportional to the field intensity.

In each sub-space spanned by  $|g, n\rangle$  and  $|e, n-1\rangle$ , the system thus performs Rabi oscillations with a slightly different frequency. If one starts with a coherent state  $|\alpha\rangle$  for the field mode, the Rabi oscillations will thus evolve at a mean frequency  $\approx g|\alpha|$ , but at large times, the oscillations will ‘get out of phase’. This leads to a ‘collapse’ of the Rabi oscillation amplitude, as illustrated in Figure 1.2. It happens on the time scale  $1/g$  which is a factor  $|\alpha|$  times longer than the period of the initial Rabi oscillations. At still larger times, of order  $1/(g|\alpha|)$ , the amplitude of the oscillations ‘revives’ again. This is due to the fact that the Rabi frequencies form a discrete, incommensurable set (the frequencies are proportional to the irrational numbers  $\sqrt{n}$ , on resonance). A more detailed analysis is presented in Sec.??.

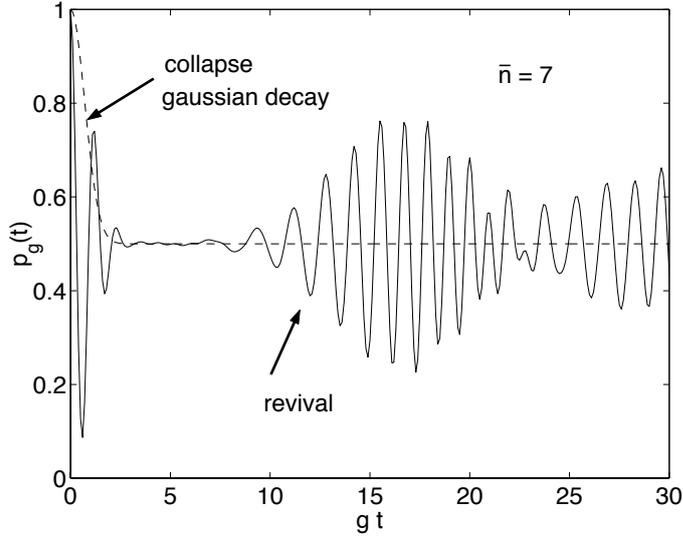


Figure 1.2: Ground state occupation  $p_g(t)$  for a two-level atom coupled to a single mode, initially in the coherent state  $|\alpha\rangle$  with  $|\alpha|^2 = 7$ . Time is in units of the ‘single-photon Rabi frequency’  $g$ .

### 1.5.3 Dissipation and Bloch equations

We now describe how the dynamics of the atomic Bloch vector is modified when so-called dissipative processes are taken into account. These processes occur because the two-level system is not closed: it is in contact with the electromagnetic field that carries away energy and information (entropy). In addition, it is subject to vacuum fluctuations (see Chapter ??). The challenge of including dissipation into quantum optics is that the equations of motion must be compatible some basic principles of quantum mechanics: states cannot evolve in an arbitrary way because probabilities remain positive, for example.

#### Spontaneous emission

As a consequence of the coupling to the quantized electromagnetic field, the excited state of the two-level atom decays by emitting a photon into an ‘empty’ mode of the electromagnetic field. This phenomenon can conveniently be described by the equations of ‘radioactive decay’ (a pair of ‘rate equations’)

$$\frac{dp_e}{dt} = -\gamma p_e, \quad \frac{dp_g}{dt} = +\gamma p_g \quad (1.57)$$

The rate  $\gamma$  gives the probability per unit time of emitting a photon and putting the atomic population from the excited state down to the ground state. The total population is conserved, as it should be for a process where the atom just changes its internal state. (In radioactive decay, ‘e’ would be a plutonium and ‘g’ an uranium atom, and the ‘photon’ an  $\alpha$ -particle.) In terms of the third component of the Bloch vector (the inversion), we have the following equation

$$\left. \frac{d}{dt} \langle \sigma_3 \rangle \right|_{\text{decay}} = -\gamma(\langle \sigma_3 \rangle + 1) \quad (1.58)$$

We also need a prescription how to take into account such a process in the dynamics of off-diagonal elements of the density matrix like  $\rho_{eg}$ . These equations cannot be chosen arbitrarily because we require that the density operator  $\rho$  remains positive under time evolution. We discuss this in more detail later in the lecture. The result is that also the dipole components of the Bloch vector decay exponentially

$$\left. \frac{d}{dt} \langle \sigma \rangle \right|_{\text{decay}} = -\Gamma \langle \sigma \rangle \quad (1.59)$$

The rate must satisfy the inequality  $\Gamma \geq \gamma/2$ , otherwise one can find initial conditions that evolve into state outside the Bloch sphere (i.e., a density matrix with negative probabilities). The process (1.59) is sometimes called “dephasing” or “decoherence” because it happens when the relative phase of a superposition state,  $\alpha|g\rangle + \beta e^{i\theta}|e\rangle$  is “diffusing” in time (with a variance that increases linearly with  $t$  like in Brownian motion). The off-diagonal elements of the density matrix are sometimes called “coherences”, they determine to what extent one has a genuine quantum superposition, distinct from a “classical” (or thermodynamical) mixture.

To compute the spontaneous decay rate  $\gamma$ , we need Fermi’s Golden Rule, a standard result from time-dependent perturbation theory. We derive this after we have learned about the quantization of the electromagnetic field, but the result is

$$\gamma = \frac{|\mathbf{d}_{ge}|^2 \omega_{eg}^3}{3\pi\epsilon_0 \hbar c^3} \quad (1.60)$$

with a typical value  $1/\gamma \sim 10$  ns for transitions in the visible range and dipole moments of the order of the Bohr magneton. We just note the scaling with the fine structure constant

$$\frac{\gamma}{\omega_{eg}} \sim \frac{e^2}{3\pi\epsilon_0 \hbar c} \frac{a_0^2}{\lambda_{eg}^2} \sim \alpha_{\text{fs}}^3 \quad (1.61)$$

On the scale of the Bohr frequencies in the atom, the decay is thus very slow.

## Bloch equations

One finds by applying the model for spontaneous decay that the average spin vector evolves according to the following set of equations, now including dissipation,

$$\frac{d}{dt}\langle\sigma\rangle = -(\mathrm{i}\omega_A + \Gamma)\langle\sigma\rangle + \mathrm{i}(\Omega/2)e^{-\mathrm{i}\omega_L t}\langle\sigma_3\rangle \quad (1.62)$$

$$\frac{d}{dt}\langle\sigma_3\rangle = -\gamma(\langle\sigma_3\rangle + 1) + \mathrm{i}\left(\Omega^* e^{\mathrm{i}\omega_L t}\langle\sigma\rangle - \Omega e^{-\mathrm{i}\omega_L t}\langle\sigma^\dagger\rangle\right) \quad (1.63)$$

This is written within the rotating wave approximation (the Hamiltonian (1.40)) but not yet in the so-called frame rotating at the laser frequency  $\omega_L$ . Using the transformation (1.41), one finds  $\sigma(t) = \tilde{\sigma}(t) e^{-\mathrm{i}\omega_L t}$  with

$$\frac{d}{dt}\langle\tilde{\sigma}\rangle = (\mathrm{i}\Delta - \Gamma)\langle\tilde{\sigma}\rangle + \mathrm{i}(\Omega/2)\langle\sigma_3\rangle \quad (1.64)$$

$$\frac{d}{dt}\langle\sigma_3\rangle = -\gamma(\langle\sigma_3\rangle + 1) + \mathrm{i}\left(\Omega^*\langle\tilde{\sigma}\rangle - \Omega\langle\sigma^\dagger\rangle\right) \quad (1.65)$$

Here, the frequencies enter via the detuning  $\Delta = \omega_L - \omega_A$  (Paris convention, some authors use the other sign).

## Rate equation limit

Assume that dephasing rate  $\Gamma$  is “faster” than all other time scales. “Adiabatic elimination of coherences” leads to

$$\langle\tilde{\sigma}\rangle_{\text{ad}} \approx \frac{\mathrm{i}(\Omega/2)\langle\sigma_3\rangle}{\Gamma - \mathrm{i}\Delta} \quad (1.66)$$

found by solving Eq.(1.64) in the steady state. *Idea*: find stationary state after an initial transient. Time scale for transient is  $1/\Gamma$ , hence short by assumption. Assume that  $\langle\sigma_3\rangle$  evolves slowly on this scale (“adiabatic following” of  $\langle\tilde{\sigma}\rangle$ ).

Insert into Eq.(1.65) for the inversion gives the *rate equation*

$$\begin{aligned} \frac{d}{dt}\langle\sigma_3\rangle &= -\gamma(\langle\sigma_3\rangle + 1) - 2 \operatorname{Im}(\Omega^*\langle\tilde{\sigma}\rangle_{\text{ad}}) \\ &= -\gamma(\langle\sigma_3\rangle + 1) - \underbrace{\frac{\Gamma|\Omega|^2}{\Gamma^2 + \Delta^2}}_{\text{absorption}}\langle\sigma_3\rangle \end{aligned} \quad (1.67)$$

The last term is the rate by which the two-level system absorbs energy from the laser (and gets excited). Indeed, for initial conditions in the ground state  $\langle\sigma_3\rangle = -1$ , one gets a positive derivative from Eq.(1.67).

**Exercise.** Calculate the time-averaged power absorbed by the two-level atom using the formula from mechanics,  $P_{\text{abs}} = \langle \dot{\mathbf{d}}(t) \cdot \mathbf{E}(t) \rangle$  by taking the time average and the quantum expectation value in the stationary state. Compare  $P_{\text{abs}}/\hbar\omega_L$  to the absorption rate in Eq.(1.67).

Rate equations are often used in condensed matter when fluorescent systems like molecules or quantum dots are embedded in a solid environment. In that case, the contact with the surrounding atoms and molecules leads to a large value for  $\Gamma$ . In this limit, the induced dipole moment is quite small (see Eq.(1.66)), and the relevant dynamics is well approximated by considering only probabilities (occupation numbers, the inversion).

## 1.5.4 A few notes on quantum dissipation

### How to generate mixed states

If a quantum system is closed and can be prepared in a pure state, then the time evolution is simply Hamiltonian,  $|\psi(t)\rangle = U(t)|\psi(0)\rangle$ , and we don't have to talk about quantum dissipation. This is not so in many settings, however.

There are a few examples how mixed (or non-pure) states arise.

**(i) Initial mixed state.** If the initial state is prepared within some probabilistic scheme, we have to work with an initial density matrix  $\rho(0) \neq |\psi(0)\rangle\langle\psi(0)|$ . This translates our incomplete knowledge about the initial conditions. Recall that density matrices can be “mixed” by forming so-called convex linear combinations

$$\rho = p\rho_1 + q\rho_2, \quad p + q = 1, \quad p, q \geq 0, \quad \text{tr } \rho_{1,2} = 1 \quad (1.68)$$

where the two density matrices  $\rho_{1,2}$  are both normalized and  $p, q$  can be interpreted as probabilities for preparing the two.

The time evolution is still simple if the system is closed (Hamiltonian evolution):

$$\rho(t) = U(t)\rho(0)U^\dagger(t) \quad (1.69)$$

or in differential form (the von-Neumann equation)

$$\frac{d}{dt}\rho = \frac{1}{i\hbar}[H, \rho] \quad (1.70)$$

A typical example is an initial state prepared with a given temperature,  $\rho(0) \propto \exp(-H_I/T)$ . Interesting dynamics then happens only if  $H_I \neq H$ .

**Exercise.** We actually don't need to solve the von Neumann equation (1.70): by expanding  $\rho(0)$  in terms of its eigenvectors, we can just evolve the two under Schrödinger's equation and mix the final states. By linearity, the result is the same.

**(ii) Reduced density matrix.** The second example is that of a “system”  $S$  coupled to another one, let's call it “bath” or “environment”  $B$ . In this setting, we restrict ourselves (*by construction*) to observables that do not give any information about the state of the environment. These observables can be written in the form  $\hat{A} \otimes \mathbb{1}_B$  where  $\mathbb{1}_B$  is the unit operator in the environment's Hilbert space. The key observation is that the expectation values for all system observables of this type can be calculated with the help of a density operator  $\rho$  for the system,

$$\langle \hat{A} \otimes \mathbb{1}_B \rangle_{S+B} = \text{tr}(\hat{A} \rho_S) \quad (1.71)$$

Note that there are many authors who do not make the distinction between  $\hat{A} \otimes \mathbb{1}_B$  and  $\hat{A}$ . The object  $\rho_S$  is called a *reduced density operator* (or matrix). It is sometimes written

$$\rho_S = \text{tr}_B \rho_{S+B} = \text{tr}_B |\psi_{S+B}\rangle \langle \psi_{S+B}| \quad (1.72)$$

where the last writing assumes that system+environment are in a pure state  $|\psi_{S+B}\rangle$ . This procedure is called “taking the partial trace” over the environment (symbolic:  $\text{tr}_B$ ), *tracing out the environment*, or “projecting into the system Hilbert space”. More precisely, the partial trace and the reduced density operator can be written in terms of the matrix elements ( $|a\rangle, |b\rangle$  are arbitrary system states)

$$\langle a | \rho_S | b \rangle = \sum_n \langle a, n | \rho_{S+B} | b, n \rangle \quad (1.73)$$

where the  $\{|n\rangle\}$  form a complete basis for the environment. You will encounter sometimes the writing

$$\text{tr}_B \rho_{S+B} = \sum_n \langle n | \rho_{S+B} | n \rangle \quad (\text{symbolic}) \quad (1.74)$$

where the object on the rhs has to be understood as having still the character of an operator in the Hilbert space of the system.

The time evolution of a system coupled to an environment produces mixed states in a dynamical way:

$$\rho_S(t) = \text{tr}_B \left[ U_{S+B}(t) \rho_{S+B}(0) U_{S+B}^\dagger(t) \right] \quad (1.75)$$

even if  $\rho_{S+B}(0)$  starts off in a pure state. This is called the “Nakajima-Zwanziger” projection. This construction is, of course, only relevant if (i) the initial state is not factorized (it is entangled) or (ii) there is some interaction between  $S$  and  $B$ . Otherwise  $U_{S+B}(t)$  factorizes, and the partial trace simply reduces to

$$\begin{aligned}
& \text{tr}_B(U_S \otimes U_B)(\rho_S \otimes \rho_B)(U_S \otimes U_B)^\dagger \\
&= \text{tr}_B(U_S \rho_S U_S^\dagger) \otimes (U_B \rho_B U_B^\dagger) \\
&= U_S \rho_S U_S^\dagger \text{tr}_B(U_B \rho_B U_B^\dagger) \\
&= U_S \rho_S U_S^\dagger
\end{aligned} \tag{1.76}$$

The Nakajima-Zwanziger projection (1.75) shares many physically interesting features and is at the basis of many generalizations of the Schrödinger equation to “open quantum systems”. The system+environment setting thus provides a conceptual framework to introduce dissipation into quantum mechanics. We shall use it in the later parts of the quantum optics course.

**(iii) Measure and forget.** This procedure of mixing states is related to the system+environment setting, but it arises from the basic postulates and can be formulated without introducing explicitly an environment. We recall the standard rule (von Neumann and Lüders) of what happens to a quantum state when an observable  $\hat{A}$  has been measured (with eigenvalue  $a$ ):

$$|\psi\rangle \mapsto |a\rangle \tag{1.77}$$

The system has “collapsed” to an eigenstate  $|a\rangle$  of the observable. This is still a pure state and corresponds to a “perfect” or projective measurement.

Now introduce probabilities and forgetting. The probability that we get the eigenvalue  $a$  is, of course, given by  $p(a) = |\langle a|\psi\rangle|^2 = \text{tr}(|a\rangle\langle a|\rho)$  where  $\rho = |\psi\rangle\langle\psi|$  for an initially pure state. Hence if we start off with a non-pure state, the von-Neumann-Lüders rule reads

$$\rho \mapsto |a\rangle\langle a| \tag{1.78}$$

In this way, we can even “purify” a mixed state! After all, the states in quantum mechanics just reflect the knowledge we have about the system.

The perfect measurement is often quite difficult to perform, however, and many states can be found that are still compatible with the measured eigenvalue  $a$ . In other words, our measurement cannot distinguish precisely among the different eigenstates  $|a\rangle$ . This is the typical scenario if the eigenvalues are continuously distributed.

Now let us imagine that we only know that we have performed the measurement “Is the system in state  $|a\rangle$ ?”, but have forgotten the result. We know that with probability  $p(a)$ , the state has collapsed (projective measurement). But with probability  $1 - p(a)$ , something else has happened. Let us assume that the state remained unchanged. By forgetting the result of the measurement, we are forced to assign to the system a mixed state:

$$\rho \mapsto (1 - p(a))\rho + p(a)|a\rangle\langle a| \quad (\text{simplest approximation}) \quad (1.79)$$

This scenario is called an “imperfect” or weak measurement. If the probability  $p(a)$  is small, the state change is also small. This is the scenario we shall use to motivate the dissipative evolution of a two-level system. An alternative notation for the probabilistic mixture of the two states can be given

$$\rho \mapsto \begin{cases} \rho & \text{with prob } 1 - p(a) \\ |a\rangle\langle a| & \text{with prob } p(a) \end{cases} \quad (1.80)$$

**Remark.** We can re-phrase this procedure within a system+environment setting. Suppose that we couple the system to an environment that can “measure” whether the system is in state  $|a\rangle$ . After some evolution time, we get an entangled state ( $|\psi\rangle$  is the initial system density state, assumed pure and  $|0\rangle$  the initial environment state)

$$|\psi, 0\rangle \mapsto \langle a|\psi\rangle|a, 1_a\rangle + U_{S+B}|\psi_\perp, 0\rangle$$

where  $|1_a\rangle$  is the (“conditional”) environment state and  $|\psi_\perp\rangle$  is the (non-normalized) system state orthogonal to  $|a\rangle$ . We construct the reduced density operator and get a mapping (between system operators)

$$|\psi\rangle\langle\psi| \mapsto \text{tr}_B(\langle a|\psi\rangle|a, 1_a\rangle + U_{S+B}|\psi_\perp, 0\rangle)(\langle a|\psi\rangle|a, 1_a\rangle + U_{S+B}|\psi_\perp, 0\rangle)^\dagger$$

Now comes the key assumption: the coupling to the environment has been sufficiently strong so that one can distinguish the environment states  $|1_a\rangle, |0\rangle$ , and the environment states contained in  $U_{S+B}|\psi_\perp, 0\rangle$ . The best we can do is that these states are orthogonal

$$\langle 1_a|0\rangle \approx 0, \quad \text{tr}_B U_{S+B}|\psi_\perp, 0\rangle\langle a, 1_a| \approx 0 \quad (1.81)$$

This removes the mixed (crossed) terms in the partial trace, and we get a mixture (with  $p(a) = |\langle a|\psi\rangle|^2$  as in QM I)

$$|\psi\rangle\langle\psi| \mapsto |a\rangle\langle a|p(a) + \text{tr}_B U_{S+B}|\psi_\perp, 0\rangle\langle\psi_\perp, 0|U_{S+B}^\dagger$$

where the first term contains the projection onto the eigenstate. The simplest assumption for the second term is that the environment does not evolve at all, provided the system is in the orthogonal state  $|\psi_\perp\rangle$ . Then  $U_{S+B}|\psi_\perp, 0\rangle \approx |\psi_\perp, 0\rangle$ , and the partial trace gives

$$|\psi\rangle\langle\psi| \mapsto |a\rangle\langle a|p(a) + \mathbb{P}_\perp|\psi\rangle\langle\psi|\mathbb{P}_\perp$$

where  $\mathbb{P}_\perp$  projects into the subspace orthogonal to  $|a\rangle$ . The last term has a trace  $1 - p(a)$ , as in Eq.(1.79), but differs slightly because of the projection. We come back to this when discussing spontaneous emission.

See the introductory article “Decoherence and the transition from quantum to classical” by Zurek (1991) for more details on this discussion. The main message is that the coupling to an environment can provide the same physics as measuring a quantum system.

### Quantum dissipation in a two-level system

Evolution over time step  $\Delta t$ . “Sufficiently small” in some sense to put together Hamiltonian evolution and measurement (“monitoring”) by an environment.

Pure Hamiltonian (for simplicity, time-independent, applies in rotating frame) ( $\hbar = 1$ )

$$\rho(t + \Delta t) \approx (\mathbb{1} - iH\Delta t)\rho(t)(\mathbb{1} + iH\Delta t) = \rho(t) - i[H\Delta t, \rho(t)] + \mathcal{O}(\Delta t^2) \quad (1.82)$$

Now observing and forgetting about the results. We consider two scenarios.

**Dephasing: measuring energy states.** We assume that with a probability  $\Delta p$ , we have been able to determine in which energy eigenstate the two-level system is. This can be achieved, for example, by performing measurements on the environment. The rule for “measure and forget” then gives (we have three outcomes)

$$\rho(t + \Delta t) = \begin{cases} \rho(t) & \text{with prob } 1 - \Delta p \\ |g\rangle\langle g| & \text{with prob } \Delta p \rho_{gg}(t) \\ |e\rangle\langle e| & \text{with prob } \Delta p \rho_{ee}(t) \end{cases} \quad (1.83)$$

This gives the mixed state, as a simple calculation shows

$$\begin{aligned} \rho(t + \Delta t) &= (1 - \Delta p)\rho(t) + \Delta p \sum_{a=g,e} \langle a|\rho(t)|a\rangle |a\rangle\langle a| \\ &= (1 - \frac{1}{2}\Delta p)\rho(t) + \frac{1}{2}\Delta p \sigma_3 \rho(t) \sigma_3 \end{aligned} \quad (1.84)$$

Concatenate the two elementary processes (1.82, 1.84) and construct an approximate time derivative

$$\frac{\Delta\rho}{\Delta t} \approx -i[H, \rho(t)] + \frac{\Delta p}{2\Delta t} \{\sigma_3 \rho(t) \sigma_3 - \rho(t)\} \quad (1.85)$$

This is the dynamical equation for a system subject to dephasing. The equation is in the so-called Lindblad form (see Eq.(1.93)), a general form for the time evolution of an open system that we shall derive later in the lecture. The rate  $\Delta p/2\Delta t$  is called the “dephasing rate”.

**Exercise.** Switch to the Heisenberg picture and calculate from Eq.(1.85) the rate of change  $\langle \Delta \sigma / \Delta t \rangle$  of the Bloch vector. Show that the non-Hamiltonian terms give

$$\left. \frac{\langle \Delta \sigma \rangle}{\Delta t} \right|_{\text{non-H}} \approx -\frac{\Delta p}{2\Delta t} \langle \sigma \rangle \quad (1.86)$$

while  $\langle \Delta \sigma_3 / \Delta t \rangle = 0$ . The monitoring of the energy levels thus does not change the inversion which is not surprising, since we made the assumption that the measured eigenstate is not changed. The dipole, that captures the relative phase of superposition states in the energy basis, however, decays with a rate  $\Delta p / 2\Delta t$ . We can thus interpret the dipole relaxation rate as the rate at which the environment acquires information about the system's energy. Note also that the decay of the dipole is the price to pay for the measurement in the energy basis – the quantum-mechanical rule that “any measurement perturbs the system” still holds.

**Spontaneous emission: quantum jumps.** The second scenario is based on the observation of the photons that a two-level atom can emit. We assume that over the evolution time  $\Delta t$ , the probability to detect an emitted photon is  $\Delta p \rho_{ee}(t)$ . We have clearly  $\Delta p = \gamma \Delta t$  according to the law of radioactive decay. In addition, once this photon has been detected, we know that the atom must be in the ground state  $|g\rangle$ . This feature is different from the previous scenario where the measurement perturbed the system in a weaker way.

Now imagine that we throw away the information that a photon has been emitted. The state then mixes into

$$\rho(t + \Delta t) = \begin{cases} \rho' & \text{with prob } 1 - \Delta p \\ |g\rangle\langle g| & \text{with prob } \Delta p \rho_{ee}(t) \end{cases} \quad (1.87)$$

where the state  $\rho'$  is normalized and corresponds to the event “no photon detected”. This can be translated into

$$\rho(t + \Delta t) = \rho'' + \Delta p \rho_{ee}(t) |g\rangle\langle g| = \rho'' + \Delta p \sigma \rho(t) \sigma^\dagger \quad (1.88)$$

where  $\rho''$  is a non-normalized state with trace  $1 - \Delta p \rho_{ee}(t)$ . The second term appearing here is called a “quantum jump”: the photon emission happens when the atom jumps from the excited to the ground state  $|e\rangle \rightarrow |g\rangle$ . The ladder (or annihilation) operator  $\sigma$  plays here a very intuitive role. If  $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$  is a pure state, the system jumps to the state  $\sigma|\psi(t)\rangle$  at the photon emission.

The first term  $\rho''$  in Eq.(1.88) now takes care of the conservation of probabilities. In our first guess (1.79) we simply took  $\rho'' = (1 - \Delta p \rho_{ee})\rho$ . This recipe must be refined here, for two reasons.

One reason is more formal: we want that  $\rho(t + \Delta t)$  to be expressed as a linear map of the state  $\rho(t)$ , but  $\Delta p \rho_{ee} \rho$  is quadratic in  $\rho$ . This reason is deeply rooted in the linearity of quantum mechanics. The Nakajima-Zwanziger scheme (1.75) which provides a very general framework for quantum dissipation, is also a linear map between density matrices.

The second reason is that the event “no photon has been detected” actually *changes our knowledge* about the system. Qualitatively speaking, it increases our confidence that the system might be in the ground state. We are having less the tendency to think it is in the excited state. After all, if the system is in the excited state, we would expect at some point a photon to appear! In the opposite limit, if over a very long time we do not detect any photons, the system must be (with a very high probability) in the ground state, and we have gained this knowledge from the sequence of “no-photon” events.

We are thus led to the following refined approximation

$$\begin{aligned}\rho'' &\approx \left(\mathbb{1} - \frac{1}{2}\Delta p |e\rangle\langle e|\right) \rho(t) \left(\mathbb{1} - \frac{1}{2}\Delta p |e\rangle\langle e|\right) \\ &\approx \rho(t) - \frac{1}{2}\Delta p \left\{ \sigma^\dagger \sigma \rho(t) + \rho(t) \sigma^\dagger \sigma \right\}\end{aligned}\quad (1.89)$$

Putting Eqs.(1.88, 1.89), together, we find the so-called “master equation” for a two-level system with spontaneous decay

$$\frac{\Delta \rho}{\Delta t} \approx -i[H, \rho(t)] + \frac{\Delta p}{\Delta t} \left\{ \sigma \rho(t) \sigma^\dagger - \frac{1}{2} \sigma^\dagger \sigma \rho(t) - \frac{1}{2} \rho(t) \sigma^\dagger \sigma \right\} \quad (1.90)$$

More details on this derivation can be found in the paper by Dalibard & al. (1992) on a “Wave-Function Approach to Dissipative Processes in Quantum Optics”.

**Exercise.** By working out matrix elements of this equation, identify  $\gamma = \Delta p / \Delta t$  with the spontaneous decay rate. The important message of this equation is that spontaneous emission changes *both* the inversion and the dipole:

$$\left. \frac{\langle \Delta \sigma \rangle}{\Delta t} \right|_{\text{non-H}} \approx -\frac{1}{2} \gamma \langle \sigma \rangle \quad (1.91)$$

$$\left. \frac{\langle \Delta \sigma_3 \rangle}{\Delta t} \right|_{\text{non-H}} \approx -\gamma (\langle \sigma_3 \rangle + 1) \quad (1.92)$$

## Lindblad master equation

Without going into the details of the derivation, we just state here that the generalized von-Neumann equations (1.85, 1.90) are special cases of a general theorem about the time evolution of a quantum system, the

**Lindblad theorem.** If a time evolution  $T_t : \rho(0) \mapsto \rho(t)$  satisfies the following conditions:

- the map  $T_t$  is linear and maps density matrices onto density matrices;
- the map  $T_t$  is completely positive<sup>3</sup>;
- expectation values evolve continuous in time;
- repeating the map corresponds to adding time lapses,  $T_t T_{t'} = T_{t+t'}$ ,

then there exists a hermitean operator  $H$  countable set of system operators  $L_k$  ( $k = 1 \dots K$ ) such that the state  $\rho(t) = T_t \rho(0)$  solves the differential equation

$$\frac{d\rho}{dt} = -i [H, \rho] + \sum_{k=1}^K \left\{ L_k \rho L_k^\dagger - \frac{1}{2} L_k^\dagger L_k \rho - \frac{1}{2} \rho L_k^\dagger L_k \right\} \quad (1.93)$$

The operators  $L_k$  are called Lindblad or jump operators.

For spontaneous emission and dephasing, only a single Lindblad operator appears, as shown in this Table:

	spont. decay	dephasing
Lindblad operator	$\sqrt{\gamma} \sigma$	$\sqrt{\kappa/2} \sigma$

where  $\kappa$  is the dephasing rate. Keeping both Lindblad operators in the time evolution, gives the Bloch equations (1.64 1.65) with a dephasing rate  $\Gamma = \kappa + \gamma/2$ . The Lindblad theorem is proven later on in the quantum optics lecture. A simple proof can be found in Nielsen & Chuang (2000) and Henkel (2007).

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<sup>3</sup>Qualitatively speaking: density matrices are mapped onto density matrices even if the system is augmented by some environment and the map  $T_t$  augmented by “nothing happens with the environment”.

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