Chapter 3

Evidence for vacuum fluctuations

Casimir energy (difference of zero-point energies), see Sec.3.1.

Electromagnetic energy density in the vacuum state, see Sec.2.3.3.

Spontaneous decay of an excited state, as calculated by Dirac (1927). The result is an exponential decay of the probability p_e of finding an atom in its excited state

$$\frac{\mathrm{d}p_e}{\mathrm{d}t} = -\gamma_e p_e, \qquad \gamma_e = \frac{|\mathbf{d}|^2 \omega_A^3}{3\pi\varepsilon_0 \hbar c^3} \tag{3.1}$$

where $\mathbf{d} = \langle g|\hat{\mathbf{d}}|e\rangle$ is the matrix element of the electric dipole operator (the "transition dipole") and ω_A the Bohr frequency.

3.1 Casimir energy

The Casimir force is the attraction between two metallic mirrors placed in vacuum. It is interpreted in terms of the change in the zero-point energy (the famous $\frac{1}{2}\hbar\omega$ of the harmonic oscillator ground state) induced by the presence of the mirrors. We give here an sketch of the calculation done by Casimir around 1948 [Proc. Kon. Ned. Akad. Wet. **51** (1948) 793].

We consider the ground state energy of the multi-mode electromagnetic field

$$E_0 = \sum_{\mathbf{k}\lambda} \frac{\hbar\omega_{\mathbf{k}\lambda}}{2}$$

that is of course infinite and compare the cases of a planar cavity formed by two mirrors (distance L) and empty space (i.e., two mirrors infinitely apart). In the first case, we have standing wave modes between the mirrors with a frequency

$$\omega^{(cav)} = c\sqrt{K^2 + k_n^2}, \qquad k_n = \frac{n\pi}{L}$$

with $K^2 = k_x^2 + k_y^2$ and $n = 1, 2, \ldots$, while in empty space,

$$\omega = c\sqrt{K^2 + k_z^2}$$

with $-\infty < k_z < \infty$. We first compute the difference in the electromagnetic mode density per volume AL where A is the 'quantization area' in the xy-plane. We cheat with the polarizations and multiply by a factor 2:

$$\begin{aligned} \rho_L(\omega) &= \frac{4\pi}{AL} \sum_{\mathbf{K},n} \delta\Big(\omega - c\sqrt{K^2 + k_n^2}\Big) \\ &= \frac{2}{L} \int_0^\infty K \mathrm{d}K \sum_n \delta\Big(\omega - c\sqrt{K^2 + k_n^2}\Big) \end{aligned} \tag{3.2}$$

The integration over K can be performed with the substitution $K \mapsto c\sqrt{K^2 + k_n^2}$ and gives

$$\rho_L(\omega) = \frac{2\omega}{Lc^2} \sum_n \Theta(\omega - ck_n) \tag{3.3}$$

where Θ is the step function. It arises because for a given n, there are no modes with frequency smaller than ck_n. The same calculation in the infinite volume gives

$$\rho_\infty(\omega) = \frac{2\omega}{c^2} \int_0^\infty \frac{\mathrm{d}k_z}{\pi} \Theta(\omega - ck_z) \tag{3.4}$$

The k_z integral can of course be performed, but we keep it here to illustrate one of the basic features of the Casimir calculation: the result originates from the difference between a 'discrete spectrum' (the sum over the k_n) and a continuum (the integral over k_z).

The Casimir energy is now found as the difference in vacuum energy *per area* in the space of length L between the mirrors:

$$\Delta E = L \int_0^\Omega \frac{\hbar\omega}{2} \left(\rho_L(\omega) - \rho_\infty(\omega)\right) \tag{3.5}$$

We have introduced an upper cutoff frequency Ω because the integrals are likely to diverge in the UV. One of the mathematical difficulties (that we are not going to discuss here) is to what extent the results depend on the cutoff. At a suitable stage of the calculation, we are going to take the limit $\Omega \to \infty$, of course.

The ω-integrals can be performed before the sum over n (the integral over k_z), and one gets

$$\Delta E = \frac{\hbar}{6\pi c^2}\left[\sum_{n=1}^{\lfloor \Omega L/\pi c\rfloor}\left(\Omega^3 - (ck_n)^3\right) - \int_0^{\Omega/c}\frac{\mathrm{d}k_z}{\pi}\left(\Omega^3 - (ck_z)^3\right)\right] \tag{3.6}$$

where $\lfloor x\rfloor$ is the largest integer smaller than x. Introducing the number $N = \Omega L/\pi c$ and the dimensionless integration variable $z = k_z L/\pi$, this can be written in the form

$$\Delta E = \frac{\hbar c\pi^2}{6L^3}\left[\sum_{n=1}^{\lfloor N\rfloor}\left(N^3 - n^3\right) - \int_0^{N}\mathrm{d}z\left(N^3 - z^3\right)\right] \tag{3.7}$$

The difference in brackets is some magic number and equal to $-1/120$ in the limit $N \to \infty$. (This is an application of the Euler-MacLaurin formula for sums. An alternative proof is sketched below.) The Casimir energy of two mirrors is thus equal to

$$\Delta E = -\frac{\hbar c\pi^2}{720L^3} \tag{3.8}$$

so that the force per unit area is $F_C/A = -\hbar c\pi^2/240L^4$: since the energy decreases as $L \to 0$, the two mirrors placed in vacuum attract each other.

Note that this result is independent of the nature of the mirrors, as well as their electric charge. The electromagnetic field only enters inasmuch as its modes give a contribution to the energy of the vacuum state. Field theorists have computed the contribution to the Casimir energy from the Dirac electron field, for example. It is small if the mirror separation is large compared to the Compton wavelength $\hbar/mc \approx 2.5\,\mathrm{pm}$ — which is nearly always the case. The Casimir energy, being attractive, is sometimes thought of a means to 'stabilize' a classical model of the electron (a bag of charge) against the Coulomb repulsion.

Sum minus integral

We use a trick in the complex plane. There is a theorem for functions f and D that are analytical in a domain limited by the integration contour $\mathcal{C}$:

$$\frac{1}{2\pi\mathrm{i}}\oint_{\mathcal{C}}\mathrm{d}z\, f(z)\frac{\mathrm{d}}{\mathrm{d}z}\log D(z) = \sum_n f(z_n) \tag{3.9}$$

where the z_n are the zeros of D in the interior of the contour. We will use $f(z) = N^3 - z^3$ and choose $D(z)$ such that it is zero for the values $z_n = n$:

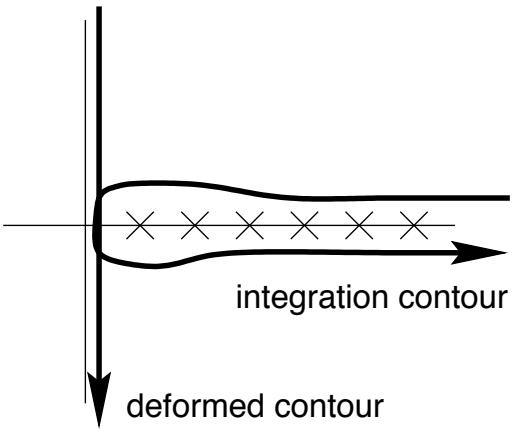

Figure 3.1: Integration contour for (3.10).

$D(z) = \sin(\pi z)$. The differentiation under the integral sign gives

$$\sum_{n=0}^{N}(N^3 - n^3) \;=\; \frac{1}{2}\oint_{\mathcal{C}_N} \mathrm{d}z\,(N^3 - z^3)\frac{\mathrm{e}^{\mathrm{i}\pi z} + \mathrm{e}^{-\mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}} \tag{3.10}$$

We chose an integration contour as shown in fig. 3.1 running from $+N$ above the real axis to 0 and going back to $+N$ below the real axis (the sum over all positive zeros of $\sin \pi z$ thus gives the sum on the left hand side). We are eventually interested in the limit $N \to \infty$. Make the following transformations on the upper and lower part of the contour:

$$\text{upper part:} \quad \frac{\mathrm{e}^{\mathrm{i}\pi z} + \mathrm{e}^{-\mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}} = -1 + \frac{2\mathrm{e}^{\mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}}$$

$$\text{lower part:} \quad \frac{\mathrm{e}^{\mathrm{i}\pi z} + \mathrm{e}^{-\mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}} = 1 + \frac{2\mathrm{e}^{-\mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}}$$

The constants ± 1 give for both the upper and lower path an integral over $N^3 - z^3$ that can be combined into

$$\sum_{n=0}^{N}(N^3 - n^3) \;=\; \int_0^N \mathrm{d}z\,(N^3 - z^3) + \oint_{\mathcal{C}} \mathrm{d}z\,(N^3 - z^3)\frac{\mathrm{e}^{\pm\mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}}$$

In the second integral, the exponential takes the appropriate sign on the upper and lower parts of the contour. The first integral on the right hand side is exactly the integral that we have to subtract in Eq.(3.7). The upper and lower parts of the contour can now be shifted onto the (positive or negative) imaginary

axis because the integrand has no singularities (these are located on the real axis only). The quarter-circle with radius $|z| = N$ contributes only a negligible amount because of the $\mathrm{e}^{\pm \mathrm{i}\pi z}$.

Choosing $z = \pm \mathrm{i}t$ on the imaginary axis, we get

$$\begin{aligned}
&\oint_{\mathcal{C}} \mathrm{d}z\, (N^3 - z^3) \frac{\mathrm{e}^{\pm \mathrm{i}\pi z}}{\mathrm{e}^{\mathrm{i}\pi z} - \mathrm{e}^{-\mathrm{i}\pi z}} \\
&= -\mathrm{i} \int_0^\infty \frac{\mathrm{d}t\, [N^3 - (\mathrm{i}t)^3]\, \mathrm{e}^{-\pi t}}{\mathrm{e}^{-\pi t} - \mathrm{e}^{\pi t}} - \mathrm{i} \int_0^\infty \frac{\mathrm{d}t\, [N^3 - (-\mathrm{i}t)^3]\, \mathrm{e}^{-\pi t}}{\mathrm{e}^{\pi t} - \mathrm{e}^{-\pi t}} \\
&= -2 \int_0^\infty \frac{\mathrm{d}t\, t^3}{\mathrm{e}^{2\pi t} - 1}
\end{aligned}$$

Note that the imaginary parts of the two integrals that involve N^3 cancel each other: we have finally eliminated the cutoff.

You have encountered the last integral in the context of blackbody radiation. Changing to the integration variable $t' = 2\pi t$, the integral gives $1/240$, so that we have in the end

$$\lim_{N\to\infty} \left(\sum_{n=0}^{N} (N^3 - n^3) - \int_0^N \mathrm{d}z\, (N^3 - z^3) \right) = -\frac{2}{240} = -\frac{1}{120} \tag{3.11}$$

as announced in the text.

3.2 Dirac's calculation of spontaeous decay

Initial state of atom and field: $|\psi(0)\rangle = |e\rangle \otimes |\mathrm{vac}\rangle = |e, \mathrm{vac}\rangle$. Ansatz for later times:

$$|\psi(t)\rangle = c_e(t)|e, \mathrm{vac}\rangle + \sum_\kappa c_\kappa(t)|g, 1_\kappa\rangle \tag{3.12}$$

where $\kappa = (\mathbf{k}, \mu)$ is a shorthand for the momentum and polarization quantum numbers of one-photon states. Time-dependent Schrödinger equation solved with perturbation theory (see Sec.1.3.2)

$$|\psi(t)\rangle \approx U(t)|\psi(0)\rangle - \frac{\mathrm{i}}{\hbar} \int_0^t \mathrm{d}t' U(t - t') H_{\mathrm{int}}(t') U(t') |\psi(0)\rangle \tag{3.13}$$

formulate in terms of vacuum correlation spectrum of electric field.

3.3 Quantum Langevin formulation

3.3.1 Idea

Many systems in quantum optics are "open" and follow equations of motion that go beyond the Schrödinger equation. This applies to systems for which the Hamiltonian is not completely known or systems in contact with an "environment". The equations of motion can be formulated as so-called master equations (similar to the Schrödinger picture, dynamics for the system density operator) or as so-called Langevin equations (similar to the Heisenberg picture, dynamics of the system operators). These equations are the quantum analogue of kinetic theories that describe the dynamics of a system including different dissipative effects. They provide, among others, the approach of the system towards thermal equilibrium, but also non-equilibrium states that are driven by external forces, the typical example being a laser field.

In quantum optics, the "system" can be an atom, a collection of atoms, or a field mode in a cavity. The system is "open" because it interacts with the "environment" (rest of the world), that is modelled as a continuum of quantized field modes. An atom decays irreversibly by emitting a photon into previously empty vacuum modes. This phenomenon of "spontaneous emission" does not allow for a description in terms of a Hamiltonian. In fact, the state of the atom does not remain pure, and entropy increases because the photon can be emitted into any direction of space. A cavity mode decays because photons escape through the cavity mirrors, effectively becoming excitations of the "modes outside the cavity". Alternatively, the field energy can be absorbed by the material making up the mirrors.

We focus first on the derivation of a master equation for a two-level atom. We shall find from the general theory two results: (1) the rate of spontaneous emission of an electronically excited state and (2) Glauber's formula for the signal of a photodetectors in terms of normally ordered correlation functions of the electric field operator.

Time scales

The derivation builds on the following hierarchy of time scales:

$$\frac{2\pi}{\omega_{\mathrm{A}}} \leq \tau_{\mathrm{c}} \ll \frac{2\pi}{\Omega} \sim \frac{1}{\gamma} \tag{3.14}$$

where the shortest time scale is the optical period of the light field that is near-resonant with the atomic transition frequency ω_{A}. This is typically smaller than a few fs (femtoseconds) (in the visible range).

The "correlation time" τ_{c} of the electromagnetic field describes qualitatively the time interval over which the quantum fluctuations of the electromagnetic field (that couple significantly to the atom) are "smooth". Two electric field measurements are not significantly "similar" if they are taken at intervals larger than τ_{c}. We give an estimate below, it turns out to be a few times larger than the optical period.

The third time scale is the Rabi period that scales with the inverse of the Rabi frequency of a laser. This depends of course on the laser power (and the atomic transition dipole), but typical values are in the 1–10 ns (nanoseconds) range. Note that this is much longer than the optical period and the field correlation time.

The last time scale is the lifetime of the excited state of the atom, inversely proportional to the spontaneous decay rate γ. This is typically in the 1–10 ns range as well, depending on the atom.

We thus have a situation similar to "Brownian motion": the atom is "shaken around" by the vacuum field fluctuations that vary much faster than the typical dynamics for the atomic state (Rabi oscillations, spontaneous decay). The master equation takes advantage of this separation of time scales to find an equation of motion that can be applied on "slow time scales". As a result, one gets a single equation that describes both, the "Hamiltonian" effects of a laser (Rabi oscillations) and the "dissipative / friction" effects due to vacuum fluctuations.

Hamiltonian

Two-level atom, field, and electric dipole interaction:

$$H = \frac{\hbar\omega_A}{2}\sigma_3 + \sum_k \hbar\omega_k a_k^\dagger a_k + \sum_k \hbar\left(g_k^* a_k^\dagger \sigma + g_k \sigma^\dagger a_k\right) \tag{3.15}$$

neglect the zero-point energy of the field. Coupling constant $\hbar g_k = -\mathbf{d}_{\mathrm{eg}} \cdot \mathbf{f}_k(\mathrm{x}_A)(\hbar\omega_k/2\varepsilon_0)^{1/2}$ with the mode function $\mathbf{f}_k(\mathrm{x})$ evaluated at the position of the atom. Sometimes called 'vacuum Rabi frequency'.

In the following, we also use the (re-scaled) electric field operator

$$E(t) = \sum_k g_k a_k(t) + \mathrm{h.c.} \tag{3.16}$$

Field correlation time

To get an estimate of the field correlation time, we compute the two-time correlation (or coherence) function

$$C(\tau) = \langle E(t+\tau)E(t)\rangle \tag{3.17}$$

If we use the coupling constants g_k to write the mode expansion of the quantized field (this is equivalent to re-scaling the electric field so that it has units of frequency), we get

$$\begin{aligned}\langle E(t+\tau)E(t)\rangle &= \sum_k |g_k|^2 \langle a_k(t+\tau)a_k^\dagger(t)\rangle \\ &= \sum_k |g_k|^2 \, \mathrm{e}^{-\mathrm{i}\omega_k\tau}\end{aligned} \tag{3.18}$$

An explicit calculation using the mode amplitudes in free space leads to an integral of the form

$$C(\tau) \sim \int_0^\infty \mathrm{d}\omega \, \omega^3 \, \mathrm{e}^{-\mathrm{i}\omega_k\tau} \tag{3.19}$$

which looks like a third derivative of a δ-function in τ. Well, not exactly since the integrations starts at $\omega = 0$. Generalizing to finite temperature, the Fourier transform of the correlation function (3.19) becomes

$$C(\omega) \sim \omega^3 \bar{n}(\omega) = \frac{\omega^3}{\exp(\hbar\omega/k_\mathrm{B}T) - 1} \tag{3.20}$$

The sign convention of the Fourier transform is chosen here such that $\omega > 0$ corresponds to photon numbers $a^\dagger a$ ('normal order'), while at $\omega < 0$, the anti-normal order $aa^\dagger$ is picked. See Fig.3.2: the spectrum is proportional to $-\omega^3$ for negative frequencies, $-\omega \gg k_\mathrm{B}T/\hbar$. Near zero frequency, it has a quadratic behaviour, at positive frequencies, a maximum near $\omega = 2k_\mathrm{B}T/\hbar$ (Wien displacement law) and an exponential decay for $\omega \gg k_\mathrm{B}T/\hbar$.

Actually, we forgot one thing in this estimate: the Hamiltonian we start with is only valid for field modes k near-resonant with the atomic transition, $\omega_k \sim \omega_\mathrm{A}$. It thus seems reasonable to restrict the frequency integral in Eq.(3.19) to an interval around ω_A, with a width $\Delta\omega$ that is typically smaller than ω_A.

Now, from the properties of the Fourier transformation, we know that the correlation function $C(\tau)$ given by the integral (3.19) has a "width in time" given

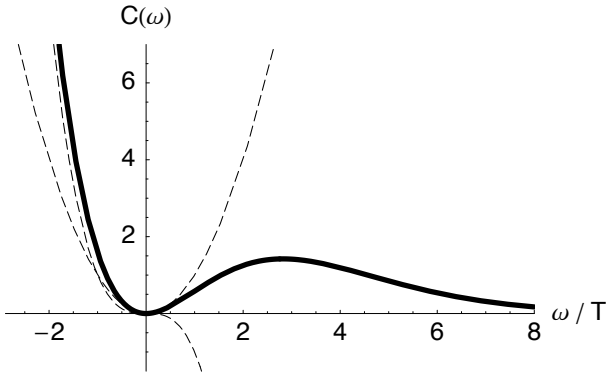

Figure 3.2: Spectrum $C(\omega)$, Eq.(3.20), of the radiation field (normal order) at finite temperature. The frequency is plotted in units of $k_{\rm B}T/\hbar$. The dashed lines are ω^2 and $-\omega^3$.

by $\Delta\tau\,\Delta\omega \sim 1$. Since this width is precisely the definition of the correlation time, we have

$$\frac{1}{\omega_{\rm A}} \leq \frac{1}{\Delta\omega} \sim \tau_{\rm c}. \tag{3.21}$$

Typical value: a few 10 fs.

3.3.2 Evolution of the atomic operators

We shall work in the Heisenberg picture. Note that operators describing different degrees of freedom (field, atom) commute at equal times.

Atomic dipole

"eliminate" the field operators by solving their equation of motion:

$$\frac{{\rm d}a_k}{{\rm d}t} = \frac{\rm i}{\hbar}\,[H,\,a_k] = -{\rm i}\omega_k a_k - {\rm i}g_k^*\sigma \tag{3.22}$$

$$a_k(t) = a_k(0)\,{\rm e}^{-{\rm i}\omega_k t} - {\rm i}g_k^* \int_0^t {\rm d}t'\,\sigma(t')\,{\rm e}^{-{\rm i}\omega_k(t-t')} \tag{3.23}$$

the "particular solution" contains the "past" of the atomic dipole operator $\sigma(t')$. Insert this into the equation for σ. Let's look at this equation first:

$$\frac{{\rm d}\sigma}{{\rm d}t} = \frac{\rm i}{\hbar}\,[H,\,\sigma] = -{\rm i}\omega_{\rm A}\sigma + {\rm i}\sum_k g_k \left[\sigma^\dagger,\,\sigma\right] a_k$$

$$= \quad -\mathrm{i}\omega_{\mathrm{A}}\sigma + \mathrm{i}\sum_k g_k\sigma_3 a_k \tag{3.24}$$

We see here that the two-level atom leads to nonlinear equations of motion: the operator product $\sigma_3 a_k$ appears. For the moment, these operators commute (at equal times). But we now want to insert the solution (3.23) for $a_k(t)$, and the two terms that appear here do not commute separately with σ_3. For this reason, we take now a specific operator order (so-called "normal order") where the annihilation operators (a_k or σ) act first. This is the order already used in Eq.(3.24), and we thus get

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = -\mathrm{i}\omega_{\mathrm{A}}\sigma + \mathrm{i}\sigma_3\mathcal{E}_0(t) + \int_0^t \mathrm{d}t' \sum_k |g_k|^2 \mathrm{e}^{-\mathrm{i}\omega_k(t-t')}\sigma_3(t)\sigma(t') \tag{3.25}$$

where we used the abbreviation

$$\mathcal{E}_0(t) = \sum_k g_k a_k(0)\,\mathrm{e}^{-\mathrm{i}\omega_k t} \tag{3.26}$$

for the freely evolving electric field operator (positive frequency component only, and re-scaled by the atomic transition dipole). In the integral over t', we see that the correlation function $C(t-t')$ appears, and using $\tau = t - t'$ as integration variable, we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = -\mathrm{i}\omega_{\mathrm{A}}\sigma + \mathrm{i}\sigma_3\mathcal{E}_0(t) + \int_0^t \mathrm{d}\tau\, C(\tau)\sigma_3(t)\sigma(t-\tau) \tag{3.27}$$

Now comes the *key observation*: under the time integral occur two very different functions. The correlation function $C(\tau)$ is very narrow in τ. The atomic operator $\sigma(t-\tau)$ contains a "fast free evolution" (generated by the first term in its equation of motion), but once this is factored out, we anticipate that its evolution is "slow":

$$\sigma(t-\tau) = \mathrm{e}^{-\mathrm{i}\omega_{\mathrm{A}}(t-\tau)}\tilde{\sigma}(t-\tau) \approx \mathrm{e}^{-\mathrm{i}\omega_{\mathrm{A}}(t-\tau)}\tilde{\sigma}(t) = \mathrm{e}^{+\mathrm{i}\omega_{\mathrm{A}}\tau}\sigma(t) \tag{3.28}$$

Here, we introduced temporarily the dipole operator $\tilde{\sigma}(t')$ (in an interaction picture).

The main benefit of this approximation is that only atomic operators at time t appear in the equation of motion. The two-level commutation rules[1] give

[1] A pedantic remark: operator products evolve as products under the Heisenberg equations of motion. This is because taking the commutator with a product is compatible with the product rule of (time) derivatives.

$\sigma_3(t)\sigma(t) = -\sigma(t)$, and we end up with

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = -\mathrm{i}\omega_\mathrm{A}\sigma + \mathrm{i}\sigma_3\mathcal{E}_0(t) - \sigma(t)\int_0^t \mathrm{d}\tau\, C(\tau)\,\mathrm{e}^{\mathrm{i}\omega_\mathrm{A}\tau} \tag{3.29}$$

The upper integration limit t is actually the difference between our initial time and the actual time where the equation of motion is computed. We now let this time difference be much larger than the correlation time τ_c. This is consistent with the assumption that the atomic dynamics is slow on the scale of the field's correlation time. Then the integrand is effectively zero at the upper limit, and we get a constant complex number

$$\gamma + \mathrm{i}\delta\omega_\mathrm{A} = \int_0^{t\gg\tau_\mathrm{c}} \mathrm{d}\tau\, C(\tau)\,\mathrm{e}^{\mathrm{i}\omega_\mathrm{A}\tau} = \frac{S(\omega_\mathrm{A})}{2} + \mathrm{i}\mathcal{P}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{S(\omega)}{\omega-\omega_\mathrm{A}} \tag{3.30}$$

where $S(\omega)$ is the Fourier transform of the correlation function $C(\tau)$ and $\mathcal{P}$ means the principal part of the integral.

Spontaneous decay rate and Lamb shift

Explicit calculation: spectrum of vacuum fluctuations

$$S(\omega) = 2\pi\sum_k |g_k|^2\delta(\omega_k-\omega) \tag{3.31}$$

Take a plane wave expansion and sum over the polarization vectors $\mathbf{u}_{\mathbf{k}\lambda}$ in the coupling constants g_k

$$\sum_\lambda |\mathbf{d}_\mathrm{eg}\cdot\mathbf{u}_{\mathbf{k}\lambda}|^2 = |\mathbf{d}_\mathrm{eg}|^2 - |\mathbf{d}_\mathrm{eg}\cdot\hat{\mathbf{k}}|^2 \tag{3.32}$$

where $\hat{\mathbf{k}}$ is the unit vector along $\mathbf{k}$. This formula arises because the $\mathbf{u}_{\mathbf{k}\lambda}$ are perpendicular to $\mathbf{k}$. Integration over the angles of $\mathbf{k}$ gives

$$\int \mathrm{d}\Omega(\hat{\mathbf{k}})\left(|\mathbf{d}_\mathrm{eg}|^2 - |\mathbf{d}_\mathrm{eg}\cdot\hat{\mathbf{k}}|^2\right) = 4\pi|\mathbf{d}_\mathrm{eg}|^2 - \frac{4\pi}{3}|\mathbf{d}_\mathrm{eg}|^2 = \frac{8\pi}{3}|\mathbf{d}_\mathrm{eg}|^2 \tag{3.33}$$

The integral over the length of $\mathbf{k}$ is trivial because of the δ-function in Eq.(3.31), its length is fixed to $|\mathbf{k}| = \omega/c$. Putting everything together, we get

$$S(\omega) = \frac{2\pi}{3\hbar^2}|\mathbf{d}_\mathrm{ge}|^2\frac{\hbar\omega}{2\varepsilon_0}\frac{8\pi\omega^2}{(2\pi c)^3} \tag{3.34}$$

where the last factor is the density of field modes per $\mathrm{d}\omega$ and volume. We can thus say that the spontaneously decaying atom is a "detector" for vacuum field fluctuations. The decay rate γ

$$\gamma = \frac{1}{2} S(\omega_{\mathrm{A}}) = \frac{|\mathbf{d}_{\mathrm{ge}}|^2 (\omega_{\mathrm{A}}/c)^3}{6\pi\hbar\varepsilon_0} \tag{3.35}$$

is also called the "natural linewidth" of the atomic transition $|\mathrm{g}\rangle \leftrightarrow |\mathrm{e}\rangle$ because it gives the width in frequency of the spontaneous emission spectrum. In order of magnitude, with $|\mathbf{d}_{\mathrm{ge}}| \sim e a_0$ (electron charge $\times$ atom size)

$$\frac{\gamma}{\omega_{\mathrm{A}}} \sim \alpha_{\mathrm{fs}} (a_0/\lambda_{\mathrm{A}})^2 \sim \alpha_{\mathrm{fs}}^3 \tag{3.36}$$

with fine structure constant $\alpha_{\mathrm{fs}} = e^2/4\pi\varepsilon_0\hbar c \approx 1/137.04$ and wavelength λ_{A} of resonant transition. Hence, indeed decay is "slow" on the scale of the optical period.

Frequency shift $\delta\omega_{\mathrm{A}}$ is related to asymmetry of vacuum spectrum around the transition frequency. Interpretation from second order perturbation theory: modes slightly below the atomic resonance, $\omega_k \leq \omega_{\mathrm{A}}$ tend to push the level $|\mathrm{e}\rangle$ upwards. Modes above resonance: push downwards. Null effect if spectrum is flat.

Actual calculation requires UV cutoff ω_{uv} and interaction Hamiltonian beyond the resonance (rotating wave) approximation (Hans Bethe $\sim$ 1948, calculation of the Lamb shift). Order of magnitude:

$$\delta\omega_{\mathrm{A}} \approx \gamma \log(\omega_{\mathrm{uv}}/\omega_{\mathrm{A}}) \tag{3.37}$$

and a consistent cutoff is the electron rest mass, $\omega_{\mathrm{uv}} \sim m_{\mathrm{e}} c^2/\hbar$. Hence comparable to the natural linewidth. Note: relativistic theory required, where wavelengths up to electron Compton wavelength $\hbar/m_{\mathrm{e}}c$ are involved. This goes beyond the "long wavelength approximation" because $\hbar/m_{\mathrm{e}}c \sim \alpha_{\mathrm{fs}} a_0 \ll a_0$.

Atomic populations

Actually, we did not yet show that γ is the rate of decay for the excited state. For this, we need the equation of motion for the occupations of the two energy levels. This is described by the atomic operator σ_3, also called the "inversion" because $\langle\sigma_3\rangle > 0$ when the excited state is more occupied than the ground state.

Heisenberg equation of motion

$$\begin{aligned}\frac{\mathrm{d}\sigma_3}{\mathrm{d}t} &= +2\mathrm{i}\sum_k \left[g_k^* a_k^\dagger \sigma - g_k \sigma^\dagger a_k\right] \\ &= 2\mathrm{i}\left[\mathcal{E}_0^\dagger(t)\sigma - \sigma^\dagger \mathcal{E}_0(t)\right] \\ &\quad - 2\int_0^t \mathrm{d}t' \left[C^*(t-t')\sigma^\dagger(t')\sigma(t) + C(t-t')\sigma^\dagger(t')\sigma(t)\right] \qquad (3.38)\end{aligned}$$

where we inserted the formal solution for $a_k(t)$ and brought the operator products in normal order. We apply to the t'-integral the same prescription as before and get

$$\begin{aligned}&\int_0^t \mathrm{d}t' \left[C^*(t-t')\sigma^\dagger(t')\sigma(t) + C(t-t')\sigma^\dagger(t')\sigma(t)\right] \\ &\approx (\gamma - \mathrm{i}\delta\omega_\mathrm{A})\sigma^\dagger(t)\sigma(t) + (\gamma + \mathrm{i}\delta\omega_\mathrm{A})\sigma^\dagger(t)\sigma(t) = \gamma(\sigma_3(t) + \mathbb{1}) \qquad (3.39)\end{aligned}$$

where the frequency shift drops out. By construction, the operator $\sigma_3(t)+\mathbb{1}$ gives the occupation of the excited state. From Eqs.(3.38, 3.39), we can thus read off the decay rate 2γ for the excited state population, while the ground state remains stable.

Finally, the equation for the inversion operator becomes

$$\frac{\mathrm{d}\sigma_3}{\mathrm{d}t} = -2\gamma(\sigma_3(t) + \mathbb{1}) + 2\mathrm{i}\left[\mathcal{E}_0^\dagger(t)\sigma - \sigma^\dagger \mathcal{E}_0(t)\right] \qquad (3.40)$$

Combined with the equation of motion for the atomic dipole operator,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = -(\gamma + \mathrm{i}\omega_\mathrm{A})\sigma + \mathrm{i}\sigma_3 \mathcal{E}_0(t) \qquad (3.41)$$

we have thus found the "optical Bloch equations".

In Eq.(3.41), we have made the replacement $\omega_\mathrm{A} + \delta\omega_\mathrm{A} \mapsto \omega_\mathrm{A}$ for the atomic frequency. This is called "renormalization": we combine the shift induced by the coupling to the vacuum field with the "naked" transition frequency into the frequency that can be physically observed. Recall that in reality, we can never "switch off" the coupling to the vacuum. Hence the "naked" two-level atom that we started with is actually a theoretical artefact.

3.3.3 Application 1: Bloch equations

Note that we did not specify yet the state of the light field: it is in fact encoded in the operator $\mathcal{E}_0(t)$ that depends on the initial field operators $a_k(0)$. Two examples will be studied now, the first one being an atom driven by a laser field. We have argued that to a good approximation, we can assume that the light field in a coherent state $|\alpha_{\rm L}\rangle$. We assume that at $t = 0$, the total system is in the product state $|\psi(0), \alpha_{\rm L}\rangle$ and take the expectation value of the Bloch equation. This gives the dynamics of the Bloch vector as follows (optical Bloch equations in the proper sense)

$$\frac{{\rm d}s}{{\rm d}t} = -(\gamma + {\rm i}\omega_{\rm A})s + ({\rm i}/2)s_3\Omega\,{\rm e}^{-{\rm i}\omega_{\rm L}t} \tag{3.42}$$

$$\frac{{\rm d}s_3}{{\rm d}t} = -2\gamma(s_3(t)+1) + {\rm i}\left[\Omega^*(t){\rm e}^{{\rm i}\omega_{\rm L}t}s - s^*\Omega{\rm e}^{-{\rm i}\omega_{\rm L}t}\right] \tag{3.43}$$

where $\Omega/2 = \langle\alpha_{\rm L}|\mathcal{E}_0(0)|\alpha_{\rm L}\rangle$ is the (complex) Rabi frequency and $\omega_{\rm L}$ the frequency of the laser mode.

These equations have time-dependent coefficients, but this can be removed by making a transformation into a "rotating frame". We make the replacement

$$s(t) \mapsto s(t){\rm e}^{-{\rm i}\omega_{\rm L}t} \tag{3.44}$$

where the "new" $s(t)$ satisfies Bloch equations with time-independent coefficients

$$\boxed{\begin{aligned}\frac{{\rm d}s}{{\rm d}t} &= -(\gamma - {\rm i}\delta)s + ({\rm i}/2)s_3\Omega \\ \frac{{\rm d}s_3}{{\rm d}t} &= -2\gamma(s_3(t)+1) + {\rm i}\left[\Omega^*s - s^*\Omega\right]\end{aligned}} \tag{3.45}$$

where $\delta = \omega_{\rm L} - \omega_{\rm A}$ is the "laser detuning". Note that the symbols are not the same throughout the books: the Rabi frequency Ω or the decay rate γ can differ by a factor of 2 (or -2), the detuning can have the opposite sign.

The Bloch equations are a "workhorse" of atomic physics and quantum optics. They are used to compute light absorption, excitation spectra, population transfer, radiation forces on atoms etc. In the exercises, you compute the stationary state of the Bloch equations (attention with the different sign for δ: this one should be correct)

$$s_{\rm ss} = \frac{-{\rm i}(\Omega/2)(\gamma + {\rm i}\delta)}{\gamma^2 + \delta^2 + \Omega^2/2} \tag{3.46}$$

$$s_{3,{\rm ss}} = \frac{-(\gamma^2 + \delta^2)}{\gamma^2 + \delta^2 + \Omega^2/2} \tag{3.47}$$

Discussion: average atomic dipole operator (induced by laser field), average inversion. Line broadening.

Exercise: total excitation N, does not commute when laser included. Expectation value of $\dot{N}$ in stationary state, interpretation as total scattered intensity.

Exercise: spectrum of spontaneous emission, from formal solution. Need formal solution of atomic dipole operator, Eq.(3.52).

3.3.4 Application 2: the Glauber photodetector

Roy Glauber (Nobel prize 2005) developped in the 1960s the theory of photodetection. His main result is that the signal of a photodetector is proportional to

$$I(t) \propto \langle E^{(-)}(t) E^{(+)}(t) \rangle \tag{3.48}$$

where $E^{(+)}(t)$ is the positive frequency part of the electric field operator at the detector position. This signal is constructed in such a way that if the field is in the vacuum state, a detector gives no signal: perfectly reasonable. But due to the presence of vacuum fluctuations (also nonzero expectation values of a product of field operators!), not easy to implement in the theory.

We can recover the relevant features of Glauber's theory with our two-level atom. We shall actually show that under suitable approximations,

$$I(t) \propto \langle \mathcal{E}_0^\dagger(t) \mathcal{E}(t) \rangle \tag{3.49}$$

where $\mathcal{E}(t)$ is the re-scaled electric field operator we introduced in Eq.(3.26). Note that it contains positive frequency components only, and also only those modes that are near-resonant with the atomic transition frequency $\omega_{\rm A}$. Indeed, Glauber's model for a photodetector is a two-state system that is prepared in the ground state. Incident light can be absorbed, leading to some population in the excited state. This population is then "rapidly removed" from the system. A physical example: the excited atom is ionized and the free electron moves away (it cannot come back to recombine into the ground state). This is actually the process that happens in a photomultiplier ("avalanche photodiode").

So let us see what our Heisenberg equations (3.40, 3.41) give when the atom in the ground state and the field is in an arbitrary state. We are interested in the rate of change of the excited state population:

$$I := \frac{\mathrm{d}p_e}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \langle \frac{\sigma_3 + \mathbb{1}}{2} \rangle = \frac{1}{2} \frac{\mathrm{d}\langle \sigma_3 \rangle}{\mathrm{d}t} \tag{3.50}$$

Since the atom is in the ground state, the operator $\sigma_3 + \mathbb{1}$ that appears in Eq.(3.40) averages to zero. We are left with

$$I = \mathrm{i}\langle \mathcal{E}_0^\dagger(t)\sigma(t) - \sigma^\dagger(t)\mathcal{E}_0(t)\rangle \tag{3.51}$$

and insert the formal solution for the atomic dipole operator [similar to Eq.(3.23)]:

$$\sigma(t) = \sigma(0)\,\mathrm{e}^{-(\gamma+\mathrm{i}\omega_\mathrm{A})t} + \mathrm{i}\int_0^t \mathrm{d}t'\,\sigma_3(t')\mathcal{E}_0(t')\,\mathrm{e}^{-(\gamma+\mathrm{i}\omega_\mathrm{A})(t-t')} \tag{3.52}$$

This gives

$$\begin{aligned} I &= -\int_0^t \mathrm{d}t' \left[\langle \mathcal{E}_0^\dagger(t)\sigma_3(t')\mathcal{E}_0(t')\rangle\,\mathrm{e}^{-(\gamma+\mathrm{i}\omega_\mathrm{A})(t-t')} + \mathrm{h.c.}\right] \\ &= \int_0^t \mathrm{d}\tau \left[\langle \mathcal{E}_0^\dagger(t)\mathcal{E}_0(t-\tau)\rangle\,\mathrm{e}^{-(\gamma+\mathrm{i}\omega_\mathrm{A})\tau} + \mathrm{h.c.}\right] \end{aligned} \tag{3.53}$$

where in the second line, we used the approximation that the inversion $\sigma_3(t')$ is evolving slowly and took its expectation value in the initial state (atom in the ground state, not correlated with the field). If we also assume that the expectation value evolves slowly with time t, $\langle \mathcal{E}_0^\dagger(t)\mathcal{E}_0(t-\tau)\rangle \approx \langle \mathcal{E}_0^\dagger(t+\tau)\mathcal{E}_0(t)\rangle$ [this is strictly true for a stationary field, but stationary fields do not give not very interesting signals on a photodetector], we can combine the '$+\mathrm{h.c.}$' term into the single integral

$$I = \int_{-t}^t \mathrm{d}t' \langle \mathcal{E}_0^\dagger(t)\mathcal{E}_0(t+\tau)\rangle\,\mathrm{e}^{-\gamma|\tau|+\mathrm{i}\omega_\mathrm{A})\tau} \tag{3.54}$$

We observe that the photodetector signal is similar to the Fourier transform of the two-time field correlation function. We can already confirm that the signal is given by a normally ordered expectation value of field operator. Let us consider two limits.

Narrow-band detector

If the decay time $1/\gamma$ is "very long" (compared to the correlation time of the field), the photodetector signal is essentially given by the field spectrum, taken at the transition frequency ω_A. Indeed, the τ-integral in Eq.(3.54) picks from the positive frequency operator $\mathcal{E}_0(t+\tau)$ those components that evolve like $\mathrm{e}^{-\mathrm{i}\omega_\mathrm{A})\tau}$.

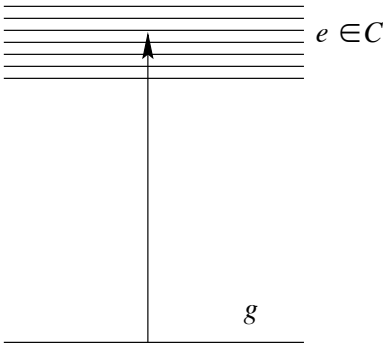

Figure 3.3: Two-level model for a fast photodetector. The signal corresponds to a sum of transition rates into all components of the upper state (quasi)continuum.

A narrow band photodetector is thus simply a 'spectrometer', and the quantity it measures is the Fourier transform of the temporal correlation (or coherence) function. Experimentally, this situation can be achieved by placing a narrow-band frequency filter in front of a photodetector.

Fast detector

In some photodetectors, the excited state has a very short lifetime $1/\gamma$. This happens for example when the excited state actually ionizes and gives away its electron. The excitation rate I is then actually an integral over all components of the excited state 'continuum', as sketched in Fig.3.3. Under the integral, the exponential $\mathrm{e}^{-\gamma|\tau|}$ becomes very short-ranged in τ, and we can make the replacement

$$\mathrm{e}^{-\gamma|\tau|} \mapsto \frac{2}{\gamma}\delta(\tau) \tag{3.55}$$

This leads to Glauber's formula

$$I(t) = \eta\langle\mathcal{E}_0^\dagger(t)\mathcal{E}_0(t)\rangle \tag{3.56}$$

where the expectation value of the instantaneous intensity operator $\mathcal{E}_0^\dagger(t)\mathcal{E}_0(t)$ appears, in normal order, of course. The prefactor η actually is a number characteristic for the detector and is called "quantum efficiency". It must be determined experimentally. If the intensity is scaled to "photons per second", then η gives the detection probability per photon, and I the "rate of detected photons per second".

Intensity (photon) correlations

Generalization to multiple detection events: $I(t)$ interpreted as probability per unit time to detect one photon at time t. Probability to detect one photon at time t_1 and another one at t_2:

$$I(t_1, t_2) = \eta^2 \langle \mathcal{E}_0^\dagger(t_1) \mathcal{E}_0^\dagger(t_2) \mathcal{E}_0(t_2) \mathcal{E}_0(t_1) \rangle, \qquad t_2 > t_1 \tag{3.57}$$

time- and normal-ordered operator product. Interpretation.

Temporal coherence

Definition in terms of factorization of $\langle \mathcal{E}_0^\dagger(t) \mathcal{E}_0(t') \rangle$ (first-order temporal coherence). And for second-order.

Examples: monochromatic field always first-order coherent. Black-body radiation not. Two-photon state is second-order coherent, but not in first-order.