Einführung in die Quantenoptik I

Wintersemester 2011/12
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Übungsaufgaben Blatt 2

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Problem 2.1 – Multipole moments and matrix elements (8 points)

The operator for the electric dipole moment of a collection of electrons is $\mathbf{d} = e \sum_\alpha \mathbf{x}_\alpha$ where e is the electron charge and $\mathbf{x}_\alpha$ are the coordinates of the electrons relative to some reference point.

(i) Write down a formula for the expectation value of $\mathbf{d}$ for a single-electron atom (like hydrogen). Show that this formula is consistent with the rules of classical electrodynamics when the probability density of the electron is interpreted as a charge distribution. [Bonus:] How does the generalization to a many-electron atom look like?

(ii) Show that the expectation value $\langle nlm|\mathbf{d}|nlm\rangle = 0$ for all wavefunctions in hydrogen. Can you generalize this result to more complicated atoms?

(iii) Calculate the z-component of the so-called electric transition dipole $\mathbf{d}_{\rm eg} = \langle \mathrm{e}|\mathbf{d}|\mathrm{g}\rangle$ in the hydrogen atom, considering the states $|\mathrm{g}\rangle = |1\mathrm{s}\rangle$, $|\mathrm{e}\rangle = |2\mathrm{p}_z\rangle$. Look up the hydrogen wave functions in your preferred quantum mechanics textbook. The order of magnitude is ea_0 [a_0: Bohr radius], of course.

(iv) For the electron momentum operator $\mathbf{p}$, one can introduce similar transition matrix elements: calculate the quantity $p_{z,\rm eg} = \langle \mathrm{e}|p_z|\mathrm{g}\rangle$. Show the relation

$$\frac{e}{m} p_{z,\rm eg} = \mathrm{i}\omega_{\rm eg} d_{z,\rm eg} \qquad (2.1)$$

where $\omega_{\rm eg} = (E_{\rm e} - E_{\rm g})/\hbar$ is the (angular) Bohr frequency for this transition.

Problem 2.2 – Relevant wavelengths (4 points)

Consider the ground state $|1\mathrm{s}\rangle$ of the hydrogen atom and look up the lowest excited states. What are the corresponding photon energies $\hbar\omega = E_{\rm e} - E_{1\rm s}$, Bohr frequencies $\omega/2\pi$, transition wavelengths $2\pi c/\omega$? (Identify suitable units.) Compare the wavelength to the size of the atom. Remember: there are excited states with higher principal quantum number, but even the electronic ground state is split into hyperfine components.

Problem 2.3 – Quantum states of a two-level system ("qubit") (8 points)

The density operator ρ for a two-level system is a hermitean 2×2 matrix with trace one. Is ρ an observable? For a system described by ρ, the expectation value of a system operator A is defined by $\langle A\rangle = \mathrm{tr}(A\rho)$ where tr is the trace.

(i) Show that this rule gives the same expectation value as the "standard prescription" for a system in the quantum state $|\psi\rangle$, when the density operator is defined as $\rho = |\psi\rangle\langle\psi|$.

(ii) Consider a density matrix of the form

$$\rho = \frac{1}{2}\left(\mathbb{1} + \sum_{j=1}^{3} s_j \sigma_j\right). \tag{2.2}$$

Show that the real numbers s_j (the components of the so-called *Bloch vector*) are equal to the expectation value of the operators σ_j. Here, $\mathbb{1}$ is the unit matrix, $\sigma_1 = \sigma + \sigma^\dagger$, $\sigma_2 = \mathrm{i}(\sigma - \sigma^\dagger)$, and σ_3 are the Pauli matrices.

(ii) The expectation value of the density matrix itself is given by $P = \operatorname{tr} \rho^2$. This real number is called the *purity* and gives us a way to distinguish between "pure states" and "mixed states". A quantum state is called *pure* iff (= if and only if) $P = 1$. Conclude that the density matrix of a pure state can be written as $\rho = |\psi\rangle\langle\psi|$. [Bonus:] this holds whatever the dimension of the system.

(iii) Show that the two-level system is in a pure state iff the Bloch vector is on the unit sphere: $\sum_j s_j^2 = 1$.