Einführung in die Quantenoptik I

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Problem 3.1 – Minimal coupling in the Coulomb gauge (5 points)

(i) Write down the minimal coupling Hamiltonian for an electron in an electromagnetic field (scalar and vector potential). Expand to lowest order in the vector potential and show that in the Coulomb gauge (i.e., $\nabla \cdot \mathbf{A} = 0$), you get as interaction Hamiltonian the so-called 'p.A coupling':

$$H_{\mathrm{pA}} = -\frac{e}{m}\mathbf{p} \cdot \mathbf{A}(\mathbf{x}) \tag{3.1}$$

where $\mathbf{x}$ and $\mathbf{p}$ are the position and momentum operators of the electron. Why is the operator ordering irrelevant here?

(ii) In the lecture, we performed a gauge transformation when going in the long-wavelength approximation from the p.A coupling to the d.E coupling,

$$\mathbf{A}'(\mathbf{x}, t) = \mathbf{A}(\mathbf{x}, t) - \nabla\chi(\mathbf{x}, t), \qquad \chi(\mathbf{x}, t) = (\mathbf{x} - \mathbf{R}) \cdot \mathbf{A}(\mathbf{R}, t) \tag{3.2}$$

Check that this transformation preserves the Coulomb gauge.

Problem 3.2 – Change the picture (10 points)

In the lecture, we have simply assumed that after a gauge transformation, one uses the new potentials in the minimal coupling Hamiltonian. But why is this so? How can a Hamiltonian be so arbitrary?

(i) Consider in a first step a 'change of picture' where a unitary transformation $S(t)$ is applied to the quantum state of a system,

$$|\psi'(t)\rangle = S(t)|\psi(t)\rangle \tag{3.3}$$

If we know that $|\psi(t)\rangle$ evolves in time under a Hamiltonian $H(t)$ and take the time derivative of Eq.(3.3), show that one finds the following Schrödinger equation for the transformed state

$$\mathrm{i}\hbar\partial_t|\psi'(t)\rangle = \left[S(t)H(t)S^\dagger(t) + \mathrm{i}\hbar\,\partial_t S(t)S^\dagger(t)\right]|\psi'(t)\rangle \tag{3.4}$$

The quantity in brackets can be identified with the 'new' Hamiltonian $H'(t)$. The object $\mathrm{i}\hbar\,\partial_t S(t)S^\dagger(t)$ is called the 'generator' of $S(t)$.

(ii) Take a constant phase factor $S(t) = \mathrm{e}^{\mathrm{i}\theta}$ and check that $H' = H$. Take a time-dependent phase, $\theta(t) = Et/\hbar$ and check that $H' = H - E$. Observe that this translates the statement: "One can always shift the energy (Hamiltonian) by an additive constant".

(iii) For a single bound electron (charge e) with position coordinate $\mathbf{x}$, take the space- and time-dependent phase $\theta(\mathbf{x}, t) = g\chi(\mathbf{x}, t)$ where χ is a smooth function and g a "coupling constant". If H is the minimal coupling Hamiltonian with potentials $\mathbf{A}$ and ϕ, check that with a suitable choice for g, H' is the Hamiltonian one gets after the gauge transformation specified by χ.

This observation is the key element of a so-called *locally* U(1) *gauge invariant theory*. The idea is so simple that instead of talking about minimal coupling, one postulates that quantum mechanics is invariant with respect to local changes in the phase of the wave function. This is only possible if the Hamiltonian contains a "gauge field" (here: the electromagnetic potentials) that transforms according to a gauge transformation when this local phase change is applied.

Problem 3.3 – Planck constant and electrodynamics (5 points)

(i) Look up the units (in SI or cgs) for the scalar potential, the vector potential and for the gauge function $\chi(\mathbf{x}, t)$ used in a gauge transformation. Check that the quantity $(e/\hbar)\chi$ is dimensionless. Check that magnetic flux Φ and $\hbar/e$ have the same units.

(ii) To get a physical interpretation for the vector potential, consider a closed path $\mathbf{x}(s)$ (s is some parameter) in configuration space and calculate the line integral

$$\oint \mathrm{d}\mathbf{x} \cdot \mathbf{A}(\mathbf{x}(s), t) \tag{3.5}$$

before and after the gauge transformation. Conclude that this integral does not change if $\chi(\mathbf{x}, t)$ is a single-valued function. Show that it is equal to the magnetic flux through the closed path.

(iii) Certain gauge transformations are singular in the sense that $\chi(\mathbf{x}, t)$ is not single-valued: imagine a graph like a helical screw (see web site), or think of the logarithm of a complex number. If a charge couples to the corresponding electromagnetic field, one can still postulate, however, that the gauge-transformed wave function $\psi'(\mathbf{x}, t) \sim \exp[-\mathrm{i}(e/\hbar)\chi(\mathbf{x}, t)]$ remains single-valued. This implies that the quantity $(e/\hbar)\chi$ can only jump in steps of 2π (from one period of the helix to the next). Now consider the vector potential $\mathbf{A}' = \nabla\chi$: the integral (3.5) then becomes an integer multiple of $\Phi = 2\pi\hbar/e$. This is the quantum of magnetic flux (well, up to a factor 2). Calculate its value ($4.14\,\mathrm{fT/m^2}$) and compare it to typical fields and loop sizes.