Einführung in die Quantenoptik I

Wintersemester 2011/12
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Problem 4.1 – Numerics and visualization (8 points)
In the lecture, we have seen the optical Bloch equations for the average dipole $\langle\sigma\rangle$ and inversion σ_3 operators (note the signs and prefactors)

$$\frac{d}{dt}\langle\sigma\rangle = (\mathrm{i}\Delta - \Gamma)\langle\sigma\rangle + \frac{\mathrm{i}}{2}\Omega\langle\sigma_3\rangle \tag{4.1}$$

$$\frac{d}{dt}\langle\sigma_3\rangle = \mathrm{i}\left(\Omega^*\langle\sigma\rangle - \Omega\langle\sigma^\dagger\rangle\right) - \gamma\left(\langle\sigma_3\rangle + 1\right) \tag{4.2}$$

(1) Take a computer and solve these equations numerically for arbitrary initial conditions. Think first about convenient units (suggestion: $1/\gamma$ for time and γ for frequency) and typical parameter values ($\gamma/2\pi \sim 10\,\mathrm{MHz}$ is typical).

(2) Visualize your solution by suitable plots. Of particular interest is (a) the comparison to the stationary solution that we found in the lecture, (b) to the rate equation limit where the dipole is "enslaved" and given by

$$\langle\sigma\rangle \approx \frac{\mathrm{i}\Omega}{2(\Gamma - \mathrm{i}\Delta)}\langle\sigma_3\rangle \tag{4.3}$$

and (c) how the Bloch vector $\langle(\sigma_1, \sigma_2, \sigma_3)\rangle$ moves in the Bloch sphere. (Recall $\sigma_1 = \sigma + \sigma^\dagger$ and $\sigma_2 = \mathrm{i}(\sigma - \sigma^\dagger)$.)

(3) Upload your results to the moodle platform and make a voting for the best visualization [up to 4 bonus points].

Problem 4.2 – Expansion of transverse vector fields (9 points)
For the quantization of the electromagnetic field, a set of vector-valued mode functions $\{\mathbf{f}_\kappa(\mathbf{x})|\kappa\}$ is essential. Check how and which of the following requirements are needed to bring the classical electromagnetic energy into the form

$$\int \mathrm{d}V \left[\frac{\varepsilon_0}{2}\dot{\mathbf{A}}^2 + \frac{1}{2\mu_0}(\boldsymbol{\nabla}\times\mathbf{A})^2\right] = \varepsilon_0 \sum_\kappa \omega_\kappa^2 \mathbf{A}_\kappa^* \cdot \mathbf{A}_\kappa \tag{4.4}$$

(1) The mode functions are orthogonal and normalized, $\int \mathrm{d}V\, \mathbf{f}_\kappa^* \cdot \mathbf{f}_\lambda = \delta_{\kappa\lambda}$.

(2) There exist a mapping $\kappa \to \bar{\kappa}$ such that $\mathbf{f}_{\bar{\kappa}} = \mathbf{f}_\kappa^*$.

(3) The expansion coefficients of the vector potential are given by

$$\mathbf{A}_\kappa(t) = \int \mathrm{d}V\, \mathbf{f}^*_\kappa(\mathbf{x}) \cdot \mathbf{A}(\mathbf{x}, t) \tag{4.5}$$

and evolve in time as $\mathbf{A}_\kappa(t) \propto \mathrm{e}^{-\mathrm{i}\omega_\kappa t}$.

(4) The mode functions satisfy the vector Helmholtz equation

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{f}_\kappa) = \varepsilon_0 \mu_0 \omega_\kappa^2 \mathbf{f}_\kappa \tag{4.6}$$

with the boundary conditions $\mathbf{n} \times \mathbf{f}_\kappa = \mathbf{0}$ on a closed surface with outward unit normal $\mathbf{n}$.

(5) The mode functions are transverse, $\boldsymbol{\nabla} \cdot \mathbf{f}_\kappa = 0$.

(6) The vector potential can be expanded as

$$\mathbf{A}(\mathbf{x}, t) = \sum_\kappa \mathbf{A}_\kappa(t) \mathbf{f}_\kappa(\mathbf{x}) \tag{4.7}$$

Check which of these requirements are satisfied by plane waves.

Problem 4.3 – Transverse δ-function (3 points)

In terms of the mode functions introduced in problem 4.2, the expansion of the unit operator (on the vector space of mode functions) reads

$$\mathbb{1} = \sum_\kappa \mathbf{f}_\kappa \otimes \mathbf{f}^*_\kappa \tag{4.8}$$

For the special case of plane waves, this becomes the so-called "transverse δ-function". In Fourier space, we have

$$\delta^\perp(\mathbf{k}) = \mathbb{1} - \frac{\mathbf{k} \otimes \mathbf{k}}{\mathbf{k}^2} \tag{4.9}$$

where $\mathbb{1}$ is a 3×3 unit matrix. In real space, one of the mappings one gets from this takes the form ($\mathbf{F}$ is a vector field with suitable properties – which?)

$$(\delta^\perp \mathbf{F})(\mathbf{x}) = \mathbf{F}(\mathbf{x}) + \boldsymbol{\nabla} \int \mathrm{d}V' \frac{\boldsymbol{\nabla}' \cdot \mathbf{F}(\mathbf{x}')}{4\pi |\mathbf{x} - \mathbf{x}'|} \tag{4.10}$$