Einführung in die Quantenoptik I

Wintersemester 2011/12
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Problem 5.1 – Lagrange formulation of electrodynamics (5 points)

In the lecture, we have stated that the Maxwell equations can be derived as Euler-Lagrange equations from the Lagrange density

$$\mathcal{L} = \frac{\varepsilon_0}{2} \left(-\boldsymbol{\nabla}\phi - \partial_t \mathbf{A}\right)^2 - \frac{1}{2\mu_0} \left(\boldsymbol{\nabla} \times \mathbf{A}\right)^2 + \mathbf{j} \cdot \mathbf{A} - \rho\phi \tag{5.1}$$

where ϕ and $\mathbf{A}$ are the scalar and vector potential, and $\mathbf{j}$ and ρ current and charge density. The Lagrange function is the space integral of $\mathcal{L}$, and the action is the integral over space and time. (i) Work out the Euler-Lagrange equation for ϕ by calculating the variation of the action with respect to ϕ and $\boldsymbol{\nabla}\phi$. Check that this gives one of the Maxwell equations. (ii) Observe that the momentum conjugate to ϕ vanishes. (iii) Calculate the change in the action when you perform a gauge transformation. (iv) [5 bonus points] Argue that $\mathcal{L}$ is a scalar (density) under Lorentz transformations.

Problem 5.2 – Playing with the mode expansion of a quantum field (5 points)

In the lecture, we have found the mode expansion for the quantized electric field operator

$$\mathbf{E}(\mathbf{x}, t) = \sum_{\kappa} \mathcal{E}(\omega_\kappa) \left[a_\kappa \boldsymbol{\varepsilon}_\kappa \, \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x} - \omega_\kappa t)} + \mathrm{h.c.} \right] \tag{5.2}$$

where $+\,\mathrm{h.c.}$ means "add the hermitean conjugate operator". The "vacuum" $|\mathrm{vac}\rangle$ is a state in the Hilbert space of the quantum field that is annihilated by all operators a_κ. Calculate the vacuum expectation values $\langle \mathrm{vac}|\mathbf{E}(\mathbf{x}, t)|\mathrm{vac}\rangle$ and $\langle \mathrm{vac}|\mathbf{d} \cdot \mathbf{E}(\mathbf{x}, t)\mathbf{d} \cdot \mathbf{E}(\mathbf{x}, t')|\mathrm{vac}\rangle$ where $\mathbf{d}$ is a constant real vector. Take the continuum limit ("quantization volume" $V \to \infty$) where appropriate.

Problem 5.3 – Vacuum energy and cosmology (5 points)

The vacuum state of the electromagnetic field has an energy density u_{vac} that depends on the cutoff momentum k_c as follows

$$u_{\mathrm{vac}} = C\hbar c k_c^4 \tag{5.3}$$

where C is a numerical constant "of order unity" (we calculate it in the lecture). (i) Fix the cutoff at the energy scale $E_{\rm GUT} = \hbar c k_c$ for the "unification" of the fundamental interactions (except gravity, look up on the Internet the keyword "grand unified theory") and make an estimate for the corresponding vacuum energy. Express this number in proton masses (times c^2) per cubic meter. (ii) Look up the length scale $\ell_{\rm Planck} = 1/k_c$ for the unification of gravity, relativity, and quantum theory and compare the corresponding vacuum energy to case (ii). (iii) Somebody told you that the density of "dark energy" in the vacuum (responsible for the accelerated expansion of the Universe, Nobel prize in physics 2011) is of the order of one proton rest mass (times c^2) per cubic meter. Find out more details on this and compare to the energy density of the cosmic microwave background.

The vacuum energy predicted by quantum electrodynamics (one of the simplest quantum field theories) is *very, very* different from the density attributed to dark energy, and there is no good explanation yet for this.

Problem 5.4 – Field operators as distributions (5 points)

We have seen the Jordan-Pauli commutator between the (transverse) fields $\mathbf{E}$ and $\mathbf{B}$ (no guarantee for signs)

$$[B_i(\mathbf{x}), E_j(\mathbf{x}')] = \frac{\mathrm{i}\hbar}{\varepsilon_0}\epsilon_{ijk}\frac{\partial}{\partial x_k}\delta(\mathbf{x}-\mathbf{x}') \tag{5.4}$$

A mathematically careful (quantum) field theory tries to make sense of the singular distribution on the right hand side. The starting point is the observation that the fields $\mathbf{A}$, $\mathbf{E}$, $\mathbf{B}$ etc are actually (operator-valued) distributions that have to be 'smeared out' with suitable smooth test functions $\mathbf{f}(\mathbf{x})$, $\mathbf{g}(\mathbf{x})$. This gives operators that behave in a less singular way; in particular you can multiply fields evaluated at the same point. Consider thus the observables

$$\mathcal{E} = \sqrt{\varepsilon_0}\int \mathrm{d}^3x\, \mathbf{f}(\mathbf{x})\cdot\mathbf{E}(\mathbf{x}), \qquad \mathcal{B} = \frac{1}{\sqrt{\mu_0}}\int \mathrm{d}^3x\, \mathbf{g}(\mathbf{x})\cdot\mathbf{B}(\mathbf{x}) \tag{5.5}$$

(i) Work out the commutator $[\mathcal{E}, \mathcal{B}]$ and conclude that orthogonal components of $\mathbf{E}$ and $\mathbf{B}$ at neighboring points cannot be measured simultaneously. Derive, as in the quantum mechanics I course, the uncertainty relation between the variances $(\Delta\mathcal{E})^2$, $(\Delta\mathcal{B})^2$.

(ii) In the construction of (ii), focus on mode functions of the form $\nabla\times\mathbf{g} = (\omega/c)\mathbf{f}$ with $\mathbf{f}$ being normalized as $\int \mathrm{d}^3x\, |\mathbf{f}(\mathbf{x})|^2 = 1$ and ω a constant. Consider a quantum state for the field where the average values of $\mathcal{E}$, $\mathcal{B}$ are zero and their variances are identical. What do you get for the 'energy' $\frac{1}{2}\langle\Delta\mathcal{E}^2\rangle + \frac{1}{2}\langle\Delta\mathcal{B}^2\rangle$?

Interpretation. One 'photon energy' $\hbar|\omega|$ is spread over the spatial support of $\mathbf{f}(\mathbf{x})$.