Einführung in die Quantenoptik I

Wintersemester 2011/12
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Übungsaufgaben Blatt 8 Ausgabe: 16. Jan 2012
Abgabe: 27. Jan 2012

Problem 8.1 – Conjugating photon operators (8 points)

You know from the solution to the Heisenberg equation of motion the following equality

$$\exp(\mathrm{i}\omega t a^\dagger a)\, a\, \exp(-\mathrm{i}\omega t a^\dagger a) = a\, \mathrm{e}^{-\mathrm{i}\omega t} \tag{8.1}$$

Use a similar "equation of motion trick" to show the following

beam splitter transformation: (8.2)

$$\exp[\mathrm{i}\omega t(b^\dagger a + a^\dagger b)]\, a\, \exp[-\mathrm{i}\omega t(b^\dagger a + a^\dagger b)] = a\, \cos(\omega t) - \mathrm{i}b\, \sin(\omega t)$$

displacement operator: (8.3)

$$\exp[\mathrm{i}t(g^* a + a^\dagger g)]\, a\, \exp[-\mathrm{i}t(g^* a + a^\dagger g)] = a - \mathrm{i}gt$$

Here, a and b are commuting mode operators, and g is a complex constant. List the hermitean conjugate equations and discuss what happens when you replace in Eq.(8.2) $b, b^\dagger \mapsto b\,\mathrm{e}^{-\mathrm{i}\theta}, b^\dagger\,\mathrm{e}^{\mathrm{i}\theta}$. Check that all these transformations preserve the commutation relations between $a, b, a^\dagger, b^\dagger$.

Problem 8.2 – Photons crossing the window (6 points)

Let's imagine that sunlight is made from green photons with a "transverse size" of approximately $50\,\mu\mathrm{m}$ (!) and a "duration" $\sim 1\,\mu\mathrm{m}/c$. Now consider this photon crossing a glass window. You know from classical electrodynamics that in a dielectric medium like glass, the energy density is

$$u = \tfrac{1}{2}\mathbf{D}\cdot\mathbf{E} + \tfrac{1}{2}\mathbf{B}^2/\mu_0 \tag{8.4}$$

where $\mathbf{D} = \varepsilon_0 n^2 \mathbf{E}$ is the displacement field and n the refractive index of glass. Find an estimate for the dipole moment per "glass molecule" induced by the photon while it is crossing the window and express it in natural units. (I find something of the order $\mathcal{O}(10^{-9})$.)

Problem 8.3 – Hong-Ou-Mandel interference (6 points)

Let a beamsplitter be described by a unitary operator S performing the mapping

$$\begin{pmatrix} a_{\rm out} \\ b_{\rm out} \end{pmatrix} = S^\dagger \begin{pmatrix} a_{\rm in} \\ b_{\rm in} \end{pmatrix} S = \begin{pmatrix} \cos\theta & -\mathrm{i}\sin\theta \\ -\mathrm{i}\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_{\rm in} \\ b_{\rm in} \end{pmatrix} \tag{8.5}$$

The operator S is given in Eq.(8.2). Show that $S|\mathrm{vac}\rangle = |\mathrm{vac}\rangle$.

Consider an incident two-photon state $|\mathrm{in}\rangle = a_{\rm in}^\dagger b_{\rm in}^\dagger |\mathrm{in}\rangle$, one photon from each input port. Calculate the output state $|\mathrm{out}\rangle = S|\mathrm{in}\rangle$.

Result: for a 50/50 beam splitter, the photons come out together ("bunched"), either in the mode to $a_{\rm out}$ or in the mode to $b_{\rm out}$. In experiments, this "bunching" can be used to check that the two photons have, apart from their input direction, exactly the same characteristics: polarization, transverse mode function. In addition, the destructive interference of the $a_{\rm out}^\dagger b_{\rm out}^\dagger |\mathrm{vac}\rangle$ state happens only when the two photons impinge on the beamsplitter within their coherence time (the "arrival time" of the photon wavepacket).