

Chapter 2

The quantized field

We give here an elementary introduction to the quantization of the electromagnetic field. We adopt the Coulomb gauge and a canonical framework. We then discuss examples for the quantum states of the field and the properties of vacuum fluctuations.

A few general remarks. The procedure followed here is also called “second quantization”. This refers, taken literally, to the quantum mechanics of a many-body system where “first quantization” corresponds to introducing operators \hat{x} , \hat{p} for a single particle (or equivalently: introducing a wave function $\psi(\mathbf{x})$). The second-quantized theory then promotes the wave function to an operator $\hat{\psi}(\mathbf{x})$. We see this procedure at work in electrodynamics where the starting point is a “classical field theory”. The result is the same: the fields $\mathbf{E}(\mathbf{x}, t)$, $\mathbf{B}(\mathbf{x}, t)$ will be replaced by operators (denoted with carets = “hats”).

In the modern language of quantum field theory, any physical object is described by a suitable field operator. Photons appear in the quantized Maxwell fields. For electrons, the Schrödinger equation is quantized (either in its relativistic form, the Dirac equation, to describe high-energy electrons, or in the non-relativistic form, to describe a many-electron system like in a solid).

There is an analogue of “first quantization” (or its “reverse” = going to the classical limit) in electrodynamics: the passage to geometrical optics.

Statements from Cohen-Tannoudji, Dupont-Roc and Grynberg. The electromagnetic field is not the wave function for the photon because the equation of motion contains a source term = photons can be created and annihilated.

You can postulate that the electromagnetic energy is the Hamiltonian of the quantum theory, with the conventional commutation relations. The Heisenberg

equations of motion then give the Maxwell–Lorentz equations. *But* this short-circuits the identification of canonically conjugate coordinates in the classical Maxwell theory (incl the freedom of gauge).

The positive-frequency part of the electromagnetic field operator is well-defined only for the free field. Sources potentially mix positive and negative frequencies.

2.1 Canonical quantization

2.1.1 Fields

Maxwell equations. We want to quantize the vacuum Maxwell equations for the electromagnetic field. With a given charge and current density, these read

$$\begin{aligned}\nabla \cdot \mathbf{B} &= 0 & \varepsilon_0 \nabla \cdot \mathbf{E} &= \rho \\ \nabla \times \mathbf{E} + \partial_t \mathbf{B} &= 0 & \nabla \times \mathbf{B} - \mu_0 \varepsilon_0 \partial_t \mathbf{E} &= \mu_0 \mathbf{j}\end{aligned}\tag{2.1}$$

Introducing the scalar and vector potentials via

$$\mathbf{B} = \nabla \times \mathbf{A}\tag{2.2}$$

$$\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A},\tag{2.3}$$

the left column of (2.1) is identically fulfilled. The Coulomb law then becomes

$$-\varepsilon_0 \nabla^2 \phi - \varepsilon_0 \nabla \partial_t \mathbf{A} = \rho\tag{2.4}$$

If we impose the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$, the vector potential drops out and the scalar potential is determined by the charge density alone:

$$-\varepsilon_0 \nabla^2 \phi = \rho\tag{2.5}$$

In this gauge, the scalar potential is thus not a dynamical degree of freedom of the field: its dynamics is ‘enslaved’ by that of the charges. This holds with suitable boundary conditions such that the homogeneous equation $\nabla^2 \phi = 0$ has no nontrivial solutions. In free space, with $\phi(\mathbf{x} \rightarrow \infty) \rightarrow 0$, we get

$$\phi(\mathbf{x}) = \frac{1}{4\pi\varepsilon_0} \int d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\tag{2.6}$$

which is a superposition of well-known Coulomb potentials.

Wave equation. We are left with the wave equation for the vector potential

$$\nabla \times (\nabla \times \mathbf{A}) + \frac{1}{c^2} \partial_t^2 \mathbf{A} = \mu_0 (\mathbf{j} - \varepsilon_0 \partial_t \nabla \phi) \equiv \mu_0 \mathbf{j}_\perp. \quad (2.7)$$

On the right hand side, we have introduced the ‘transverse current’. Its divergence is zero because of Eq.(2.5) and charge conservation:

$$\nabla \cdot \mathbf{j}_\perp = \nabla \cdot \mathbf{j} + \partial_t \rho = 0. \quad (2.8)$$

The name ‘transverse’ comes from the fact that in spatial Fourier components, the current $\mathbf{j}_\perp(\mathbf{k})$ must be perpendicular to \mathbf{k} . One also says that the vector potential in the Coulomb gauge is transverse because $\nabla \cdot \mathbf{A} = 0$. The transversality of the source term in (2.7) ensures that if \mathbf{A} is transverse at one time, it is also transverse at all later times.

Conservation laws. Charge: $\partial_t \rho + \nabla \cdot \mathbf{j} = 0$.

Energy (Poynting theorem):

$$\partial_t u + \nabla \cdot \mathbf{S} = -\mathbf{j} \cdot \mathbf{E} \quad (2.9)$$

$$u = \frac{\varepsilon_0}{2} \mathbf{E}^2 + \frac{1}{2\mu_0} \mathbf{B}^2 \quad \text{energy density} \quad (2.10)$$

$$\mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \quad \text{Poynting vector} \quad (2.11)$$

Momentum and angular momentum.

2.1.2 Plane wave expansion

The canonical quantization scheme starts from the classical theory in its Hamilton formulation. We first identify the field and its canonically conjugate momentum. For a classical single particle, position \mathbf{x} and $\mathbf{p} = \partial L / \partial \dot{\mathbf{x}}$ from the Lagrangian do the job. The Lagrangian of electrodynamics is discussed in the exercises and contains the vector potential \mathbf{A} as coordinate and gives as canonical momentum the electric field (\mathcal{L} is the Lagrangian density)

$$\mathbf{\Pi} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{A}}} = -\varepsilon_0 \mathbf{E} \quad (2.12)$$

We next have to check (see exercises) that the following expression generates the Maxwell equations by the canonical equations

$$H = \int dV \left(\frac{1}{2\varepsilon_0} \mathbf{\Pi}^2 + \frac{1}{2\mu_0} (\nabla \times \mathbf{A})^2 \right) \quad (2.13)$$

This is not surprising since we are dealing with the volume integral over the electromagnetic energy density. Note that we quantize only the free field here (no charges or currents).

In classical Hamiltonian electrodynamics, the fact that \mathbf{A} and $\mathbf{\Pi}$ are canonically conjugate is expressed by the following Poisson bracket

$$\{A_i(\mathbf{x}), \Pi_j(\mathbf{x}')\} = -\delta_{ij}^\perp(\mathbf{x} - \mathbf{x}') \quad (2.14)$$

where $\delta_{ij}^\perp(\mathbf{x} - \mathbf{x}')$ is the so-called transverse δ -function. It acts like the unit operator on the space of transverse vector fields \mathbf{F} :

$$F_i(\mathbf{x}) = \sum_j \int d^3x' \delta_{ij}^\perp(\mathbf{x} - \mathbf{x}') F_j(\mathbf{x}') \quad (2.15)$$

while vector fields that are not transverse are projected into the subspace of transverse fields. The Poisson bracket in Eq.(2.14) is an integral over (functional derivatives) with respect to the vector potential and the conjugate momentum field. More details follow in Section 2.4.

Diagonalize the field energy. In the field Hamiltonian (2.13), the field value $\mathbf{A}(\mathbf{x})$ is coupled to its neighboring values by the derivative. The Hamiltonian is therefore not in “diagonal form”. But it can be brought into one by expanding the field in suitable mode functions, with plane waves being a natural choice. We note that this expansion has nothing to do with quantum field theory: it is done already at the classical level (when “eigenmodes” of a laser cavity are calculated, for example). The names “quantization volume” and “quantized wave vector” are therefore ill chosen. Don’t confuse this with the next step where really quantization happens (see Section ??).

The complex character of plane waves, combined with the polarization of the field, leads to some technical difficulties as we see in a moment. We take plane waves in a box of volume V with periodic boundary conditions, and expand (the same expansion applies to the momentum field $\mathbf{\Pi}$)

$$\mathbf{A}(\mathbf{x}) = \sum_{\mathbf{k}} \mathbf{A}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \quad (2.16)$$

where the wave vectors \mathbf{k} are discrete and have a spacing of the order $V^{-1/3}$. Some simple manipulations then bring the Hamiltonian in the form

$$H = V \sum_{\mathbf{k}} \left(\frac{1}{2\varepsilon_0} \mathbf{\Pi}_{\mathbf{k}} \cdot \mathbf{\Pi}_{-\mathbf{k}} + \frac{\varepsilon_0 c^2 k^2}{2} \mathbf{A}_{\mathbf{k}} \cdot \mathbf{A}_{-\mathbf{k}} \right). \quad (2.17)$$

We have used that for the Fourier amplitudes of transverse fields, $\mathbf{k} \cdot \mathbf{A}_{\mathbf{k}} = 0$ and the expression for the speed of light. We recognize here the dispersion relation in free space $\omega_{\mathbf{k}}^2 = k^2 / \epsilon_0 \mu_0 = c^2 k^2$.

This already decomposes into a sum over the wave vectors \mathbf{k} . Only waves with \mathbf{k} and $-\mathbf{k}$ are coupled. The complex number $A_{\mathbf{k}\mu}$ therefore encodes two degrees of freedom that are related to the plane waves with wave vectors \mathbf{k} and $-\mathbf{k}$.

The main task of quantization is to replace the fields by operators such that the Poisson bracket becomes the commutator. Referring to Eq.(2.14), we introduce the plane wave expansion of the transverse δ -function:

$$\delta_{ij}^{\perp}(\mathbf{x} - \mathbf{x}') = \sum_{\mathbf{k}\mu} \frac{e_{\mathbf{k}\mu i} e_{\mathbf{k}\mu j}^*}{V} e^{i\mathbf{k} \cdot (\mathbf{x} - \mathbf{x}')} \quad (2.18)$$

where the polarization vectors $\mathbf{e}_{\mathbf{k}\mu}$ are normalized (there are two basis vectors with $\mathbf{k} \perp \mathbf{e}_{\mathbf{k}\mu}$). We have taken complex polarization vectors here, this is useful for circular polarization and to discuss the spin of the electromagnetic field.

2.1.3 Field operators

Now, quantization. The Poisson bracket $\{\cdot, \cdot\}$ is replaced by a commutator $(i/\hbar)[\cdot, \cdot]$. For the field and its conjugate momentum, we therefore get

$$i \left[\hat{A}_i(\mathbf{x}), \hat{\Pi}_j(\mathbf{x}') \right] = -\hbar \delta_{ij}^{\perp}(\mathbf{x} - \mathbf{x}') \quad (2.19)$$

You know from quantum mechanics I that this commutator provides uncertainty relations. Recalling that $\mathbf{\Pi}$ is the electric field, we therefore find that it cannot be measured simultaneously with the vector potential at the same position. The situation is actually worse because the transverse δ -function is not local in space (see the exercises).

When we insert the mode expansions for \mathbf{A} and $\mathbf{\Pi}$ in the commutator (2.19), we get a (double) sum over \mathbf{k}, μ . Comparing each term in the sum to the plane-wave expansion of the transverse δ -function [Eq.(2.18)], we get

$$i \left[\hat{A}_{\mathbf{k}i}, \hat{\Pi}_{\mathbf{k}'j} \right] = -\frac{\hbar}{V} \delta_{\mathbf{k}+\mathbf{k}',0} \sum_{\mu} e_{\mathbf{k}\mu i} e_{\mathbf{k}\mu j}^* \quad (2.20)$$

We implement this commutator by introducing annihilation and creation operators $a_{\mathbf{k}\mu}, a_{\mathbf{k}\mu}^{\dagger}$, see Eqs.(2.28, 2.29) below. Before we look at this expression, let us step back and simplify the problem.

Reminder: harmonic oscillator. Let us consider canonical coordinates x and p with the Hamiltonian (a simplified version of (2.17))

$$\hat{H} = \frac{V}{2\varepsilon_0} \hat{p}^2 + \frac{V\varepsilon_0\omega_k^2}{2} \hat{x}^2 \quad (2.21)$$

We also simplify the commutation relation (2.19) into

$$i[\hat{x}, \hat{p}] = -\frac{\hbar}{V} \quad (2.22)$$

To implement this, we try the definitions

$$\hat{x} = \sqrt{\frac{\hbar}{2V}} \lambda (a + a^\dagger) \quad (2.23)$$

$$\hat{p} = \sqrt{\frac{\hbar}{2V}} \frac{1}{i\lambda} (a - a^\dagger) \quad (2.24)$$

This choice is motivated by the fact that both \hat{x} and \hat{p} are hermitean operators. Note that the scale factor λ is not specified by the commutation relation. That one is implemented if we take

$$-\frac{\hbar}{V} \stackrel{!}{=} i[\hat{x}, \hat{p}] = \frac{\hbar}{2V} [\hat{a} + \hat{a}^\dagger, \hat{a} - \hat{a}^\dagger] = -\frac{\hbar}{V} [\hat{a}, \hat{a}^\dagger] \implies [\hat{a}, \hat{a}^\dagger] = 1 \quad (2.25)$$

The operators \hat{a} and \hat{a}^\dagger are known as ladder operators from the quantum mechanics I lecture. They are clearly dimensionless, the factor λ can hence be used to give \hat{x} and \hat{p} the correct physical dimension. More fundamentally, it provides a way to simplify (in fact: minimize) the Hamiltonian (2.21). Inserting Eqs.(2.23, 2.24), we get

$$\hat{H} = \frac{\hbar}{4} \left(-\frac{1}{\varepsilon_0\lambda^2} (\hat{a} - \hat{a}^\dagger)^2 + \varepsilon_0\omega_k^2\lambda^2 (\hat{a} + \hat{a}^\dagger)^2 \right) \quad (2.26)$$

The terms with a^2 and $a^{\dagger 2}$ drop out from this difference if we choose $\varepsilon_0\lambda^2 = 1/\omega_k$. This gives (here, the anticommutator is denoted by $\{\cdot, \cdot\}_+$)

$$\hat{H} = \frac{\hbar\omega_k}{2} \{\hat{a}, \hat{a}^\dagger\}_+ = \frac{\hbar\omega_k}{2} (2\hat{a}^\dagger\hat{a} + 1) \quad (2.27)$$

We are going to see that the full electromagnetic Hamiltonian has a similar structure, with just a sum over wave vectors and polarizations to be added.

Interpretation: photons and particles. The physical interpretation of this Hamiltonian in ordinary quantum mechanics is well known: the operator $\hat{a}^\dagger \hat{a}$ has integer eigenvalues $n = 0, 1, 2, \dots$. One has a ground state $|0\rangle$ with energy $\frac{1}{2}\hbar\omega_k$ and excited states with an equidistant spacing $\hbar\omega_k$ in energy.

This interpretation is essentially changed in the quantum field theory: the energy levels are interpreted as the presence of particles (“photons”) in a given mode of the field, each particle carrying an energy $\hbar\omega_k$. The classical mode frequency is thus translated into an energy: note how the scale factor \hbar postulated by de Broglie arises from the commutator between the quantum operators. The equidistant energy levels are a consequence of our assumption that the particles (photons) are not interacting with each other. The ladder operators of QM I can now be interpreted literally as “creation” and “annihilation” operators and implement mathematically the non-conservation of particles. Example: the decay process of particle “b” into a pair of particles “a” ($b \rightarrow a+a$) is implemented by an operator product $a^\dagger a^\dagger b$ in the Hamiltonian. This kind of products of more than two operators are called interaction Hamiltonians: they will be treated systematically in perturbation theory (and graphically represented by Feynman diagrams).

Creation and annihilation operators

For the full quantum field, we modify the rules (2.23, 2.24) for the canonical operators to include the polarization label

$$\hat{\mathbf{A}}_{\mathbf{k}} = \sqrt{\frac{\hbar}{2\varepsilon_0\omega_k V}} \sum_{\mu} \left(\hat{a}_{\mathbf{k}\mu} \mathbf{e}_{\mathbf{k}\mu} + \hat{a}_{-\mathbf{k}\mu}^\dagger \mathbf{e}_{-\mathbf{k}\mu}^* \right) \quad (2.28)$$

$$\hat{\mathbf{\Pi}}_{\mathbf{k}} = \sqrt{\frac{\hbar\varepsilon_0\omega_k}{2V}} (-i) \sum_{\mu} \left(\hat{a}_{\mathbf{k}\mu} \mathbf{e}_{\mathbf{k}\mu} - \hat{a}_{-\mathbf{k}\mu}^\dagger \mathbf{e}_{-\mathbf{k}\mu}^* \right) \quad (2.29)$$

These expressions are constructed to satisfy the “hermiticity condition” $(\hat{\mathbf{A}}_{\mathbf{k}})^\dagger = \hat{\mathbf{A}}_{-\mathbf{k}}$ that replaces conventional hermiticity in the simpler example before. The annihilation and creation operators are postulated to satisfy the following commutation relations (the generalization of (2.25)):

$$\left[\hat{a}_{\mathbf{k}\mu}, \hat{a}_{\mathbf{k}'\mu'} \right] = 0, \quad \left[\hat{a}_{\mathbf{k}\mu}, \hat{a}_{\mathbf{k}'\mu'}^\dagger \right] = \delta_{\mathbf{k}\mathbf{k}'} \delta_{\mu\mu'} \quad (2.30)$$

The commutator $\left[\hat{A}_{\mathbf{k}i}, \hat{\Pi}_{\mathbf{k}'j} \right]$ now gets contributions only from the ‘mixed term’ $\left[\hat{a}_{\mathbf{k}\mu}, \hat{a}_{-\mathbf{k}\mu}^\dagger \right]$ that appears twice. Putting the prefactors together, we find

$$i \left[\hat{A}_{\mathbf{k}i}, \hat{\Pi}_{\mathbf{k}'j} \right] = -\frac{\hbar}{2V} \delta_{\mathbf{k},-\mathbf{k}'} \sum_{\mu} \left\{ e_{\mathbf{k}\mu i} e_{\mathbf{k}\mu j}^* + e_{-\mathbf{k}\mu i}^* e_{-\mathbf{k}\mu j} \right\} \quad (2.31)$$

The polarization vectors can be constructed in such a way that $\mathbf{e}_{-\mathbf{k}\mu}^* = \mathbf{e}_{\mathbf{k}\mu}$ (this works with real vectors or with circular polarization vectors, see the exercises). Then the two terms in curly brackets combine, and we get exactly what we asked for in Eq.(2.20).

2.1.4 The observables of quantum electrodynamics

Field Hamiltonian

We have now all elements together that we need for field quantization. In the following, the results are illustrated by working out the basic observables of the theory: the field Hamiltonian and the field operators, written as mode expansions.

To get the Hamiltonian, we have to add one physical consideration, however. In which order do we take the operator products that replace the classical products $\mathbf{A}^2(\mathbf{x})$ or $\mathbf{A}_{\mathbf{k}} \cdot \mathbf{A}_{-\mathbf{k}}$? After all, the Fourier operators need not commute. (And indeed, they don't, as Eq.(2.28) shows.) The operator ordering has fundamental consequences for the vacuum energy, as we shall see. There are two conventional choices

— *symmetric order* $|\mathbf{A}_{\mathbf{k}}|^2 \mapsto \frac{1}{2} \sum_i \{ \hat{A}_{\mathbf{k}i}, \hat{A}_{-\mathbf{k}i} \}_+$

This has the advantage that the Hamilton operator is hermitean by construction.

— *normal order* $|\mathbf{A}_{\mathbf{k}}|^2 \mapsto \sum_i : \hat{A}_{\mathbf{k}i} \hat{A}_{-\mathbf{k}i} :$

Here, the notation $: \dots :$ denotes a re-arrangement of annihilation and creation operators in such a way that the annihilation operators act first:

$$: a_{\mathbf{k}\mu} a_{\mathbf{k}'\mu'}^\dagger : = a_{\mathbf{k}'\mu'}^\dagger a_{\mathbf{k}\mu}. \quad (2.32)$$

This has the advantage that in the ground state of the quantum field (all modes have zero excitations), $\langle 0 | : \hat{H} : | 0 \rangle = 0$. The ground state therefore has zero energy, and we get rid of potentially divergent zero-point fluctuations. Note that for the normal ordering scheme, we have to know already what is the ground state.

Let us start with the symmetrically ordered Hamiltonian. Inserting the field operators into the mode expansion (2.17), one gets

$$\hat{H} = \sum_{\mathbf{k}} \frac{\hbar\omega_{\mathbf{k}}}{8} (-) \sum_{\mu\mu'} \left\{ \hat{a}_{\mathbf{k}\mu} \mathbf{e}_{\mathbf{k}\mu} - \hat{a}_{-\mathbf{k}\mu}^\dagger \mathbf{e}_{-\mathbf{k}\mu}^*, \hat{a}_{-\mathbf{k}\mu'} \mathbf{e}_{-\mathbf{k}\mu'} - \hat{a}_{\mathbf{k}\mu'}^\dagger \mathbf{e}_{\mathbf{k}\mu'}^* \right\}_+$$

$$+ \sum_{\mathbf{k}} \frac{\hbar\omega_{\mathbf{k}}}{8} \sum_{\mu\mu'} \left\{ \hat{a}_{\mathbf{k}\mu} \mathbf{e}_{\mathbf{k}\mu} + \hat{a}_{-\mathbf{k}\mu}^\dagger \mathbf{e}_{-\mathbf{k}\mu}^*, \hat{a}_{-\mathbf{k}\mu'} \mathbf{e}_{-\mathbf{k}\mu'} + \hat{a}_{\mathbf{k}\mu'}^\dagger \mathbf{e}_{\mathbf{k}\mu'}^* \right\}_+ \quad (2.33)$$

As in our simple example before, the products of “like” operators $\hat{a}_{\mathbf{k}\mu} \hat{a}_{-\mathbf{k}\mu'}$ drop out of this difference. For the “mixed” products, we observe the orthogonality of the polarization vectors,

$$\mathbf{e}_{\mathbf{k}\mu} \cdot \mathbf{e}_{\mathbf{k}\mu'}^* = \mathbf{e}_{-\mathbf{k}\mu}^* \cdot \mathbf{e}_{-\mathbf{k}\mu'} = \delta_{\mu\mu'} \quad (2.34)$$

which brings us to a single sum over polarizations:

$$\hat{H} = \sum_{\mathbf{k}\mu} \frac{\hbar\omega_{\mathbf{k}}}{4} \left(\left\{ \hat{a}_{\mathbf{k}\mu}, \hat{a}_{\mathbf{k}\mu}^\dagger \right\}_+ + \left\{ \hat{a}_{-\mathbf{k}\mu}^\dagger, \hat{a}_{-\mathbf{k}\mu} \right\}_+ \right) \quad (2.35)$$

The second term gives, after the summation, the same result as the first term because the mode frequency $\omega_{\mathbf{k}}$ does not depend on the sign of \mathbf{k} . Hence, we get a sum over mode as a simple generalization of Eq.(2.27):

$$\hat{H} = \sum_{\mathbf{k}\mu} \frac{\hbar\omega_{\mathbf{k}}}{2} (2\hat{a}_{\mathbf{k}\mu}^\dagger \hat{a}_{\mathbf{k}\mu} + 1) \quad (2.36)$$

We discuss the structure of the Hilbert space (the eigenstates of \hat{H}) in Sec.2.3.2. But the ground state is a transparent concept already at this stage. It is denoted $|0\rangle$ or $|\text{vac}\rangle$ and is defined by the state annihilated by *all* annihilation operators,

$$\hat{a}_{\mathbf{k}\mu} |\text{vac}\rangle = 0, \quad \text{for all } \mathbf{k}, \mu \quad (2.37)$$

This state is called the “vacuum” because obviously, all excitation numbers are zero: there are no particles. Obviously, the vacuum state is an eigenstate of the field Hamiltonian (2.36) with an energy

$$\hat{H} |\text{vac}\rangle = E_{\text{vac}} |\text{vac}\rangle, \quad E_{\text{vac}} = \sum_{\mathbf{k}\mu} \frac{\hbar\omega_{\mathbf{k}}}{2} \quad (2.38)$$

This (divergent) vacuum energy is discussed in the exercises. We touch here one of the big open problems in fundamental physics: there are several reasonable cutoffs that can be taken to make this sum convergent, none of them gives physically reasonable results. Observable consequences of the vacuum energy (despite it being divergent, one can extract finite and measurable quantities) are discussed in Secs.2.3.3, 3.2.

Exercise. Show that the Heisenberg equation of motion for the mode operators $a_{\mathbf{k}\mu}$ has the solution

$$a_{\mathbf{k}\mu}(t) = a_{\mathbf{k}\mu} e^{-i\omega_k t} \quad (2.39)$$

Since this looks like the solution to the time-dependent Schrödinger equation, this is called a “positive frequency” operator. The creation operator $a_{\mathbf{k}\mu}^\dagger(t)$ is a “negative frequency” operator.

Mode expansion of field operators

We now come back to the operator for the vector potential. We combine the Fourier operators Eq.(2.28) with the mode expansion to get

$$\hat{\mathbf{A}}(\mathbf{x}, t) = \sum_{\mathbf{k}\mu} \sqrt{\frac{\hbar}{2\varepsilon_0\omega_k V}} \left(\hat{a}_{\mathbf{k}\mu}(t) \mathbf{e}_{\mathbf{k}\mu} + \hat{a}_{-\mathbf{k}\mu}^\dagger(t) \mathbf{e}_{-\mathbf{k}\mu}^* \right) e^{i\mathbf{k}\cdot\mathbf{x}} \quad (2.40)$$

In the second term, we can use $-\mathbf{k}$ as wave vector index. It is then easy to see that this produces the hermitean conjugate of the first term, bringing the field operator into a manifestly hermitean form:

$$\hat{\mathbf{A}}(\mathbf{x}, t) = \sum_{\mathbf{k}\mu} \sqrt{\frac{\hbar}{2\varepsilon_0\omega_k V}} \left(\hat{a}_{\mathbf{k}\mu} \mathbf{e}_{\mathbf{k}\mu} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega_k t} + \text{h.c.} \right) \quad (2.41)$$

This is called the mode expansion of the quantized electromagnetic field. The rotation of the vector potential gives us the mode expansion of the magnetic field operator. It involves the polarization vectors $\mathbf{k} \times \mathbf{e}_{\mathbf{k}\mu} = (\omega_k/c) \mathbf{b}_{\mathbf{k}\mu}$ where for circular polarizations $\mathbf{b}_{\mathbf{k}\mu}$ is actually proportional to $\mathbf{e}_{\mathbf{k}\mu}$. With linear polarizations, one gets a vector orthogonal to the electric field, of course.

The electric field operator is found from the conjugate momentum field $\hat{\mathbf{E}} = -\hat{\mathbf{\Pi}}/\varepsilon_0$ and its mode expansion is

$$\hat{\mathbf{E}}(\mathbf{x}, t) = \sum_{\mathbf{k}\mu} \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}} i \left(\hat{a}_{\mathbf{k}\mu} \mathbf{e}_{\mathbf{k}\mu} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega_k t} - \text{h.c.} \right) \quad (2.42)$$

$$\hat{\mathbf{B}}(\mathbf{x}, t) = \sum_{\mathbf{k}\mu} \sqrt{\frac{\hbar\omega_k \mu_0}{2V}} i \left(\hat{a}_{\mathbf{k}\mu} \mathbf{b}_{\mathbf{k}\mu} e^{i\mathbf{k}\cdot\mathbf{x} - i\omega_k t} - \text{h.c.} \right) \quad (2.43)$$

It is a useful check that the relation $\mathbf{E} = -\partial_t \mathbf{A}$ also holds for the quantum fields.

These mode expansions are the basic “work horse” for quantum optics. To memorize the formulas, a simple shortcut: the quantum theory works with the same “mode functions” (here: plane waves) as in classical electrodynamics. The electric field energy density for each mode, $\frac{1}{2}\epsilon_0|\mathbf{E}|^2$, is equal to $\frac{1}{2}\hbar\omega_k/V$ (sum the squares of the two terms in the brackets), times the photon number $\hat{a}_{\mathbf{k}\mu}^\dagger\hat{a}_{\mathbf{k}\mu}$. The magnetic energy density gives the same contribution so that one photon carries an energy density $\hbar\omega_k/V$. (This is for a plane wave mode delocalized over the volume V , of course.)

Field commutators, second turn

It is an instructive exercise to work out that the mode expansions (2.41, 2.42, 2.43) yield the following commutation relations between the field operators (all of them follow from the basic commutator (2.19)):

$$i \left[\hat{A}_i(\mathbf{x}), \hat{E}_j(\mathbf{x}') \right] = \frac{\hbar}{\epsilon_0} \delta_{ij}^\perp(\mathbf{x} - \mathbf{x}') \quad (2.44)$$

$$i \left[\hat{B}_i(\mathbf{x}), \hat{E}_j(\mathbf{x}') \right] = -\frac{\hbar}{\epsilon_0} \epsilon_{ijk} \partial_k \delta(\mathbf{x} - \mathbf{x}') \quad (2.45)$$

where in the second equation, we have made use of the fact that under the rotation $\nabla \times$, there is no difference between the normal and the transverse δ -function.

Interaction with matter

If we keep the interaction with matter in the description, there is one additional term in the Hamiltonian (density), $-\mathbf{j} \cdot \mathbf{A}$. To be precise, we need in the Coulomb gauge the transverse current \mathbf{j}^\perp – in this way the theory becomes gauge invariant with respect to gauge transformation allowed under the Coulomb gauge ($\mathbf{A}' = \mathbf{A} + \nabla\chi$ with $\nabla^2\chi = 0$).

The interaction Hamiltonian with the transverse current takes the following form in the quantized theory:

$$H_{\text{int}} = - \int d^3x \mathbf{j}^\perp(\mathbf{x}) \cdot \mathbf{A}(\mathbf{x}) \quad (2.46)$$

$$= - \sum_{\mathbf{k}\mu} \sqrt{\frac{\hbar}{2\epsilon_0\omega_k V}} \left\{ a_{\mathbf{k}\mu} \mathbf{e}_{\mathbf{k}\mu} \cdot \mathbf{j}_{\mathbf{k}}^{\perp*} + \text{h.c.} \right\}, \quad (2.47)$$

where $\mathbf{j}_{\perp,\mathbf{k}}$ is the spatial Fourier transform. It is, of course, also possible to work with the electric dipole interaction. The gauge transformation that leads to this

formulation can be read as a unitary transformation in the quantized field theory (see the book by Cohen-Tannoudji & al. (1987) for a discussion).

2.2 Alternative formulations

- with continuous momentum (Fourier integrals).
- with general mode functions (Sec.2.4.4)
- a bit more of functional analysis (functional derivatives etc)
- the full matter+field quantum theory, incl relativistic matter

Continuous wave vectors. Mode expansion of electric field operator

$$\mathbf{E}(\mathbf{x}, t) = \int d^3k \sqrt{\frac{\hbar\omega_k}{2\varepsilon_0(2\pi)^3}} \sum_{\mu} (\mathbf{e}_{\mathbf{k}\mu} a_{\mathbf{k}\mu} e^{i\mathbf{k}\cdot\mathbf{x}} + \text{h.c.}) \quad (2.48)$$

We read off the replacement rule by comparison to (2.42): substitute $V \mapsto (2\pi)^3$ and replace the summation over \mathbf{k} by an integral. This changes, of course, the dimensions of the annihilation operators. The commutation relation (2.30) becomes

$$[\hat{a}_{\mathbf{k}\mu}, \hat{a}_{\mathbf{k}'\mu'}^{\dagger}] = \delta(\mathbf{k} - \mathbf{k}') \delta_{\mu\mu'} \quad (2.49)$$

The interpretation is now that the operator

$$d\hat{N} = \hat{a}_{\mathbf{k}\mu}^{\dagger} \hat{a}_{\mathbf{k}\mu} d^3k \quad (2.50)$$

gives the photon number in an element d^3k of the wavevector space. The total photon number and the field energy are, for instance, given by

$$\hat{N} = \int d^3k \sum_{\mu} \hat{a}_{\mathbf{k}\mu}^{\dagger} \hat{a}_{\mathbf{k}\mu}, \quad \hat{H} = \int d^3k \sum_{\mu} \hbar\omega_k \hat{a}_{\mathbf{k}\mu}^{\dagger} \hat{a}_{\mathbf{k}\mu} \quad (2.51)$$

while they are sums in a finite quantization volume.

Quantization of wave packets. We can also associate a bosonic operator to a ‘‘wavepacket’’. Consider two orthogonal, transverse vector fields $\mathbf{f}(\mathbf{k})$ and $\mathbf{g}(\mathbf{k})$ and construct

$$\begin{aligned} \hat{a} &= \int d^3k \sum_{\mu} \mathbf{f}(\mathbf{k}) \cdot \mathbf{e}_{\mathbf{k}\mu} a_{\mathbf{k}\mu} \\ \hat{b} &= \int d^3k \sum_{\mu} \mathbf{g}(\mathbf{k}) \cdot \mathbf{e}_{\mathbf{k}\mu} a_{\mathbf{k}\mu} \end{aligned} \quad (2.52)$$

Then the commutator is

$$\begin{aligned}
[\hat{a}, \hat{b}^\dagger] &= \int d^3k d^3k' \sum_{\mu\mu'} \mathbf{f}(\mathbf{k}) \cdot \mathbf{e}_{\mathbf{k}\mu} \mathbf{g}^*(\mathbf{k}') \cdot \mathbf{e}_{\mathbf{k}'\mu'}^* [a_{\mathbf{k}\mu}, a_{\mathbf{k}'\mu'}] \\
&= \int d^3k \sum_{\mu} \mathbf{f}(\mathbf{k}) \cdot \mathbf{g}^*(\mathbf{k}) = 0
\end{aligned} \tag{2.53}$$

where we get the L^2 scalar product in the space of transverse vector fields. If the vector field \mathbf{f} is normalized in the corresponding L^2 -norm, we get $[\hat{a}, \hat{a}^\dagger] = 1$. The operator $\hat{a}^\dagger \hat{a}$ thus counts the photon number in the “wave packet” described by \mathbf{f} . These kinds of wave packets play the role of the normalizable mode functions of the simple formulation with a finite quantization volume. They are useful to describe few-photon pulses that are localized spatially and spread over some frequency range.

2.3 Photons and the quantum vacuum

2.3.1 ‘Photons’

The present quantized description of the electromagnetic field allows us to give a more precise meaning to the word ‘photon’. *A photon is an excitation of a mode of the field.* We have seen that the quantized field reduces to a collection of harmonic oscillators, one for each mode. As we know from the harmonic oscillator, its stationary states are labelled by non-negative numbers $n = 0, 1, \dots$. One says that in these states, the mode contains ‘ n photons’. The creation operator a^\dagger whose action on these states is $a^\dagger |n\rangle \propto |n+1\rangle$, thus ‘creates one photon’. This picture is consistent with the assumption (dating back to Einstein (1905)) that photons correspond to ‘energy packets’ of $\hbar\omega_k$ of the electromagnetic field. If a plane wave mode expansion is used, we can also say that the momentum of a photon is $\hbar\mathbf{k}$, as we know from de Broglie (1926) or from the Compton effect.

It is however possible to use different mode expansions for the same field. For example, we could have used an expansion in terms of spherical vector harmonics which differs from the plane wave expansion by a (infinite-dimensional) unitary transformation. A single-photon state in the plane wave basis thus becomes a superposition of single-photon states in infinitely many spherical modes. Conversely, a ‘photon’ in this description would not correspond to a plane wave (its momentum would not be $\hbar\mathbf{k}$), but it would have a definite angular momentum with respect to the origin. It is even possible to define photons that are wavepackets localized in time, by superposing plane waves with neighboring frequencies. This picture allows to describe experiments with ‘single-photon

pulses'. We refer to the exercises to look at these properties in more detail.

Finally, we precall that the computation of a suitable set of mode functions is a 'classical' problem: no quantum theory is needed to state it. Using a cubic 'quantization' box, one can show that the plane waves with wave vectors $\mathbf{k} = (2\pi/L)(n_x, n_y, n_z)^T$ ($n_i \in \mathbb{Z}$) are orthogonal with respect to the scalar product

$$\int_{L^3} d^3x \mathbf{f}^*(\mathbf{x}) \cdot \mathbf{g}(\mathbf{x}). \quad (2.54)$$

This is clear for different wave vectors, $\mathbf{k} \neq \mathbf{k}'$. But for a given \mathbf{k} , one can also find two orthogonal polarization vectors $\boldsymbol{\varepsilon}_{1,2}$ that give orthogonal modes (the scalar product $\boldsymbol{\varepsilon}_1^* \cdot \boldsymbol{\varepsilon}_2$ is zero). The corresponding magnetic field mode functions are also orthogonal. The electrodynamics becomes a 'quantum' theory only when the amplitudes of the mode functions become suitably normalized operators.

2.3.2 The Fock-Hilbert space

The Hilbert space of the quantized field is constructed from the mode operators a_κ and a_κ^\dagger . The state of lowest energy is called the 'vacuum state', $|\text{vac}\rangle$ or $|0\rangle$:

$$a_\kappa |\text{vac}\rangle = 0 \quad (2.55)$$

No confusion between the normalizable state $|0\rangle$ and the zero vector 0 in the Hilbert space. This is why we often prefer to write $|\text{vac}\rangle$.

The 'one-photon sector' is spanned by the infinitely many states with one photon per mode

$$|1_\kappa\rangle = a_\kappa^\dagger |\text{vac}\rangle \quad (2.56)$$

...and so on. A typical state can be labelled by its 'occupation numbers' $|n_1, \dots, n_\kappa, \dots\rangle = |\{n_\kappa\}\rangle$, it contains n_κ photons in the mode κ . The so-called Fock-Hilbert space is generated by taking linear combinations of these basis vectors with complex coefficients. From linear combinations with a finite number of terms, one completes the space by limiting procedures with respect to a suitable topology (related to the usual scalar product). Since the field theory ultimately contains a infinitely many modes and even a continuum, the topological structure can be quite intricate.

The field operators (vector potential, electric and magnetic fields) act between the N - and $N \pm 1$ -photon sectors via the ladder operators a_κ and a_κ^\dagger they contain. Their expectation values in the state $|\{n_\kappa\}\rangle$ are thus zero. To get a nonzero expectation value, one must construct superpositions of number states

with different particle numbers. These states are not stationary in general. In quantum optics, an example of such states are the coherent states, useful to describe classical fields or to approximate a laser field. In high-energy physics, one usually discards such superpositions by a ‘super-selection rule’: one argues that for massive particles, the rest mass is so large that the relative phase between the components that differ in particle number varies so rapidly that one cannot distinguish, in practice, between a superposition state and a mixed state (to be described by a density matrix, see below).

2.3.3 Vacuum fluctuations

In the vacuum state, the expectation value $\langle \mathbf{E}(\mathbf{x}, t) \rangle = \langle \text{vac} | \mathbf{E}(\mathbf{x}, t) | \text{vac} \rangle = 0$ because a_κ annihilates the vacuum state and a_κ^\dagger can be made to act to the left on the vacuum state which is annihilated. **Note.** The same is true for any stationary (or number) state, see Sec. 2.3.2.

The vacuum fluctuations become visible in the next moment of the field: for a given mode κ ,

$$\frac{\hbar\omega_\kappa}{2\varepsilon_0} \langle (\mathbf{f}_\kappa(\mathbf{x})a(t) + \text{h.c.})^2 \rangle = \frac{\hbar\omega_\kappa}{2\varepsilon_0} |\mathbf{f}_\kappa(\mathbf{x})|^2 \langle a_\kappa(t)a_\kappa^\dagger(t) \rangle = \frac{\hbar\omega_\kappa}{2\varepsilon_0} |\mathbf{f}_\kappa(\mathbf{x})|^2 \quad (2.57)$$

in the last step, we have used that $a_\kappa(t) = a_\kappa(0) e^{-i\omega_\kappa t}$. In the case of plane wave modes, we have $|\mathbf{f}_\kappa(\mathbf{x})|^2 = 1/V$. The sum over wave vectors \mathbf{k} and polarization indices λ can be written in the form

$$\langle \mathbf{E}^2(\mathbf{x}, t) \rangle = \int_0^\infty \frac{d\omega}{2\pi} \frac{\hbar\omega}{2\varepsilon_0} \rho(\omega) \quad (2.58)$$

where $\rho(\omega)$ is the so-called ‘local density of modes’ (per unit frequency and unit volume):

$$\rho(\mathbf{x}; \omega) = 2\pi \sum_\kappa |\mathbf{f}_\kappa(\mathbf{x})|^2 \delta(\omega - \omega_\kappa) = \frac{2\pi}{V} \sum_\kappa \delta(\omega - \omega_\kappa) \quad (2.59)$$

For the plane wave modes, ω_κ only depends on the magnitude of \mathbf{k} , and in the continuum limit,

$$\sum_{\mathbf{k}} = V \int \frac{d^3k}{(2\pi)^3} \quad (2.60)$$

we get after a simple integration

$$\rho(\omega) = \frac{2}{\pi} \frac{\omega^2}{c^3} \quad (2.61)$$

The full vacuum fluctuation of the electric field is then infinite

$$\langle \mathbf{E}^2(\mathbf{x}, t) \rangle = \frac{\hbar}{\pi \varepsilon_0 c^3} \int_0^\infty \frac{d\omega}{2\pi} \omega^3 \quad (2.62)$$

because the integral diverges at the upper limit. This correlates with an infinite electromagnetic energy density in vacuum (multiply with $\varepsilon_0/2$ and add the magnetic component, which doubles the result). This “infinite vacuum energy” is one of the unresolved problems in physics. “Intuitive cutoffs” at short wavelengths, for example, at the Planck scale (10^{-35} m) give a finite energy density, but with a value that differs by something like 120 orders of magnitude from the energy density associated with cosmological observations (including “dark energy”, “cosmological constants” and so on). A simple discussion of the vacuum energy problem has been given by Adler & al. (1995).

Exercise. Find a cutoff such that the vacuum energy density equals the ‘critical density of the Universe’ (the critical mass density is roughly 10^{-29} g/cm³, given the current expansion rate of the Universe.)

A finite value can be found, if we compute an autocorrelation function of the electric field. A similar calculation gives

$$\langle \mathbf{E}(\mathbf{x}, t) \cdot \mathbf{E}(\mathbf{x}, t') \rangle = \int_0^\infty \frac{d\omega}{2\pi} \frac{\hbar \omega}{2\varepsilon_0 c^3} \rho(\omega) e^{-i\omega(t-t')} \quad (2.63)$$

This leads to the integral representation of the Γ -function in the complex plane and finally to

$$\langle \mathbf{E}(\mathbf{x}, t) \cdot \mathbf{E}(\mathbf{x}, t') \rangle = \frac{3\hbar}{\pi^2 \varepsilon_0 c^3 \tau^4} \quad (2.64)$$

which is finite for all $\tau = t - t' \neq 0$. For any finite value of τ , the electric vacuum energy density is thus of the order of $\hbar \omega_\tau / \lambda_\tau^3$ with the characteristic frequency $\omega_\tau = 1/\tau$ and wavelength $\lambda_\tau = c\tau$.

2.4 Second turn

We give in this section a few more technical details on field quantization. The information contained here is not part of the ordinary curriculum of this year.

2.4.1 Transverse δ -function

We have encountered the ‘transverse δ -function’ $\delta_{ij}^\perp(\mathbf{x} - \mathbf{x}')$ in the commutator (2.19) between the vector potential and its canonically conjugate momentum. This is a distribution that acts

like a δ -function on fields with zero divergence, and projects an arbitrary vector field \mathbf{F} on its transverse part \mathbf{F}^\perp

$$F_l^\perp(\mathbf{x}) = \int d^3x' \delta_{lm}^\perp(\mathbf{x} - \mathbf{x}') F_m(\mathbf{x}') \quad (2.65)$$

By definition, the transverse part has zero divergence, $\nabla \cdot \mathbf{F}^\perp = 0$. The projection is most easily constructed in Fourier space

$$F_j^\perp(\mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} \frac{k^2 \delta_{jl} - k_j k_l}{k^2} \tilde{F}_l(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} \quad (2.66)$$

where $\tilde{\mathbf{F}}(\mathbf{k})$ is the spatial Fourier transform of $\mathbf{F}(\mathbf{x})$. This relation allows to deduce explicit expressions for the transverse δ -function.

Note that in the wave equation (2.7) for the vector potential, we have an alternative relation between the current density \mathbf{j} and its transverse part:

$$\mathbf{j}_\perp = \mathbf{j} - \varepsilon_0 \partial_t \nabla \phi,$$

The ‘longitudinal part’ that is subtracted here is thus related to the electrostatic field created by the corresponding charge density. More details are discussed in the exercises.

2.4.2 Matter

Let us consider that all matter is made from charged point particles with charges e_α and positions \mathbf{r}_α . The electric charge and current densities are then given by

$$\rho(\mathbf{x}, t) = \sum_\alpha e_\alpha \delta(\mathbf{x} - \mathbf{r}_\alpha(t)), \quad \mathbf{j}(\mathbf{x}, t) = \sum_\alpha e_\alpha \mathbf{v}_\alpha(t) \delta(\mathbf{x} - \mathbf{r}_\alpha(t)) \quad (2.67)$$

The sum runs over all the particles. Charge conservation is ensured provided that $\dot{\mathbf{r}}_\alpha = \mathbf{v}_\alpha$.

The point charges are thus the ‘sources’ for the electromagnetic field. But their motion is also influenced by the fields via the Newton-Lorentz equations:

$$\frac{d}{dt} \frac{m_\alpha \mathbf{v}_\alpha}{\sqrt{1 - v_\alpha^2/c^2}} = e_\alpha (\mathbf{E}(\mathbf{r}_\alpha) + \mathbf{v}_\alpha \times \mathbf{B}(\mathbf{r}_\alpha)) \quad (2.68)$$

We use a relativistic framework here, and with the Lorentz factor, the time derivative is actually the one for the relativistic particle momentum. Note that in these equations, the coordinate \mathbf{r}_α enters generally in a nonlinear way. This differs from the Maxwell equations that are linear in the fields and potentials and whose solutions are linear in the charge and current distributions. (The superposition principle can be applied, see Eq.(2.6).) All nonlinear effects in optics can ultimately be traced back to the nonlinear response of matter to an applied electromagnetic field.

Quantization. The theory outlined so far describes physics at the end of the 19th century. It is unable to describe a stable state of matter because of the ‘radiation catastrophe’: positive and negative charges circle around each other, radiate electromagnetic waves and lose energy. Stable bound states exist only when the particle dynamics is quantized—recall the hydrogen atom.

Let us recall the typical energy and length scales for the hydrogen atom. The energy levels in Hydrogen are given by

$$E_n = -\frac{\text{Ryd}}{n^2} = -\frac{me^4}{2(4\pi\epsilon_0)^2\hbar^2} \frac{1}{n^2} \quad (2.69)$$

where the Rydberg constant is $1 \text{ Ryd} \approx 13.6 \text{ eV}$. (In cgs units, drop the factor $(4\pi\epsilon_0)^2$.) The size of the Hydrogen atom is of the order of the Bohr radius

$$a_0 = \frac{4\pi\epsilon_0 \hbar^2}{me^2} \quad (2.70)$$

The typical wavelength of an electromagnetic wave resonant with a transition in Hydrogen is therefore of the order of

$$\lambda = \frac{\hbar c}{\text{Ryd}} = \frac{2}{\alpha_{\text{fs}}} a_0 \quad (2.71)$$

$$\frac{1}{\alpha_{\text{fs}}} = \frac{4\pi\epsilon_0 \hbar c}{e^2} \approx 137 \quad (2.72)$$

Here, α_{fs} is the fine structure constant. Its inverse can be understood as a measure of the speed of light in ‘atomic units’ (the natural units for the Hydrogen problem). The value $1/\alpha_{\text{fs}} \approx 137$ is fairly large. This means two things:

- the size of the hydrogen atom is small compared to the wavelength of resonant radiation: the Lorentz force (2.68) has therefore only a weak dependence on \mathbf{r}_α ;
- the typical velocity of an electron in the Hydrogen atom is in the non-relativistic regime: we can therefore use non-relativistic mechanics to describe the matter response.

This will justify several approximations for the atom-light interaction that we are going to make in the rest of the lecture.

2.4.3 Lagrange-Hamilton formulation

Lagrangian. We now proceed to quantize the wave equation (2.7) in the ‘canonical way’. The first step is to guess the corresponding Lagrangian. A good guess is the Lagrangian

$$L = -\sum_{\alpha} m_{\alpha} c^2 \sqrt{1 - \dot{\mathbf{r}}_{\alpha}^2/c^2} + \int d^3x \mathcal{L}_{F+I} \quad (2.73)$$

where the Lagrangian density for the field and its interaction with the electric charges is given by

$$\mathcal{L}_{F+I} = \frac{\epsilon_0}{2} \left(\dot{\mathbf{A}} + \nabla\phi \right)^2 - \frac{1}{2\mu_0} (\nabla \times \mathbf{A})^2 - \phi\rho + \mathbf{A} \cdot \mathbf{j}. \quad (2.74)$$

Via the Euler-Lagrange equations, one gets the Maxwell equations (2.5,2.7) for the fields and the Newton-Lorentz equation (2.68) for the particles. Note that the Lagrangian (2.73,2.74) is invariant under gauge transformations

$$\phi \mapsto \phi - \partial_t \chi, \quad \mathbf{A} \mapsto \mathbf{A} + \nabla \chi \quad (2.75)$$

where $\chi(\mathbf{x}, t)$ is an arbitrary smooth function. This gauge invariance is connected to charge conservation.

Coulomb gauge. We now proceed to specialize to the Coulomb gauge, simplify the Lagrangian and derive the Hamiltonian. We start with the terms involving the scalar potential in the Lagrangian (2.74). The mixed term is

$$\dot{\mathbf{A}} \cdot \nabla \phi = \nabla \cdot (\dot{\mathbf{A}} \phi) - \phi \nabla \cdot \dot{\mathbf{A}} \quad (2.76)$$

The first term is a divergence, and leads to a surface integral when integrated over the volume. We adopt the usual boundary condition that at the (infinitely remote) surface, the fields vanish: then this term is zero. The second term is zero in the Coulomb gauge $\nabla \cdot \mathbf{A} = 0$.

The term quadratic in the scalar potential is

$$(\nabla \phi)^2 = \nabla \cdot (\phi \nabla \phi) - \phi \nabla^2 \phi = \nabla \cdot (\phi \nabla \phi) + \phi \rho / \varepsilon_0 \quad (2.77)$$

using the Laplace equation (2.5). The second term thus combines with the interaction part $-\phi \rho$ in the Lagrangian that becomes $-\frac{1}{2} \phi \rho$. This energy can be interpreted as the Coulomb interaction energy between the charges:

$$V_{\text{Coul}} = \frac{1}{2} \int d^3x \phi \rho = \frac{1}{2} \int d^3x d^3x' \frac{\rho(\mathbf{x}) \rho(\mathbf{x}')}{4\pi \varepsilon_0 |\mathbf{x} - \mathbf{x}'|} \quad (2.78)$$

$$= \frac{1}{2} \sum_{\alpha, \beta} \frac{e_\alpha e_\beta}{4\pi \varepsilon_0 |\mathbf{r}_\alpha - \mathbf{r}_\beta|} \quad (2.79)$$

where the factor $\frac{1}{2}$ ensures that all pairs of charges are only counted once. The divergent self-interaction for $\mathbf{r}_\alpha = \mathbf{r}_\beta$ also appears here. It is usually discarded. *The key point to note is that in the Coulomb gauge, the contribution of the scalar potential depends only on the particle coordinates. It is not a proper degree of freedom of the fields.*

To summarize, in the Coulomb gauge, the Lagrangian can be split into the following form:

$$L = - \sum_{\alpha} m_{\alpha} c^2 \sqrt{1 - v_{\alpha}^2 / c^2} - V_{\text{Coul}}(\{\mathbf{r}_{\alpha}\}) + \int d^3x \mathcal{L}_{\text{F+I}}^{\perp} \quad (2.80)$$

with

$$\mathcal{L}_{\text{F+I}}^{\perp} = \frac{\varepsilon_0}{2} \dot{\mathbf{A}}^2 - \frac{1}{2\mu_0} (\nabla \times \mathbf{A})^2 + \mathbf{A} \cdot \mathbf{j}. \quad (2.81)$$

We have added the subscript \perp to remind ourselves that this is only valid if the vector potential is transverse.

Hamiltonian. For the Hamiltonian, we need the canonical momenta conjugate to \mathbf{r}_{α} and \mathbf{A} :

$$\mathbf{p}_{\alpha} = \frac{\partial L}{\partial \dot{\mathbf{r}}_{\alpha}} = \frac{m \dot{\mathbf{r}}_{\alpha}}{\sqrt{1 - \dot{\mathbf{r}}_{\alpha}^2 / c^2}} + e_{\alpha} \mathbf{A}(\mathbf{r}_{\alpha}) \quad (2.82)$$

$$\mathbf{\Pi}(\mathbf{x}) = \frac{\delta L}{\delta \dot{\mathbf{A}}(\mathbf{x})} = \varepsilon_0 \dot{\mathbf{A}}(\mathbf{x}) \quad (2.83)$$

The particle momentum contains the relativistic kinetic momentum and an electromagnetic contribution. For the field momentum, it looks as if ε_0 were the 'mass' and $\dot{\mathbf{A}}$ the velocity. But the derivative looks strange, it is a ...

Functional derivative. In the field case, we have a continuous collection of degrees of freedom, labelled by the space-points \mathbf{x} . This requires a generalization of the notion of a derivative to the infinite-dimensional case: the ‘functional derivative’ $\delta L/\delta \dot{\mathbf{A}}(\mathbf{x})$. It is the generalization of a gradient.

In mathematical terms, if we have a ‘functional’ $L[\mathbf{A}(\mathbf{x})]$, i.e. a mapping from the space of vector fields into the real numbers, its functional derivative is defined by the following approximation:

$$L[\mathbf{A}(\mathbf{x}) + \delta \mathbf{A}(\mathbf{x})] \approx L[\mathbf{A}(\mathbf{x})] + \int d^3x \left. \frac{\delta L}{\delta \mathbf{A}(\mathbf{x})} \right|_{\mathbf{A}(\mathbf{x})} \delta \mathbf{A}(\mathbf{x}) + \mathcal{O}(\delta \mathbf{A}^2) \quad (2.84)$$

Here, the second line is an example of a linear functional because the integral is linear in $\delta \mathbf{A}(\mathbf{x})$. The function with which the small deviation $\delta \mathbf{A}(\mathbf{x})$ is weighted under the integral *defines* the functional derivative. A mathematical theorem ensures that in a suitable space of functions, all linear functionals take this integral form.

It is a simple exercise to derive with (2.84) expression (2.83) for the field momentum $\mathbf{\Pi}$. We shall return to a less trivial example below.

The Hamiltonian is given by

$$H = \sum_{\alpha} \mathbf{p}_{\alpha} \cdot \dot{\mathbf{r}}_{\alpha} + \int d^3x \mathbf{\Pi} \cdot \dot{\mathbf{A}} - L \quad (2.85)$$

where the field part is the obvious generalization to a continuous set of degrees of freedom. Putting everything together, we get

$$H = \sqrt{[\mathbf{p}_{\alpha} - e_{\alpha} \mathbf{A}(\mathbf{r}_{\alpha})]^2 c^2 + m^2 c^4} + V_{\text{Coul}}(\{\mathbf{r}_{\alpha}\}) + \int d^3x \left[\frac{\mathbf{\Pi}^2}{2\epsilon_0} + \frac{(\nabla \times \mathbf{A})^2}{2\mu_0} \right] \quad (2.86)$$

where the interaction between matter and (transverse) field arises due to the ‘minimal coupling’ prescription only. (The term linear in \mathbf{j} in the Lagrangian cancels with the term linear in $\dot{\mathbf{r}}_{\alpha}$ in (2.85).)

Canonical equations. The motion of particles and fields in the Hamiltonian formalism can be described in a compact way in terms of Poisson brackets. We discuss this in some detail because they provide another example of functional derivatives and because they bear strong similarities to the commutators of the quantum theory. In addition, it turns out to be tricky to get the transverse wave equation (2.7).

The Poisson bracket provides the time evolution of any function Q (or functional) of the coordinates and momenta by

$$\dot{Q} = \{H, Q\} \quad (2.87)$$

where we define

$$\{A, B\} = \sum_{\alpha} \frac{\partial A}{\partial \mathbf{p}_{\alpha}} \cdot \frac{\partial B}{\partial \mathbf{r}_{\alpha}} - \frac{\partial A}{\partial \mathbf{r}_{\alpha}} \cdot \frac{\partial B}{\partial \mathbf{p}_{\alpha}} + \int d^3x \left[\frac{\delta A}{\delta \mathbf{\Pi}(\mathbf{x})} \cdot \frac{\delta B}{\delta \mathbf{A}(\mathbf{x})} - \frac{\delta A}{\delta \mathbf{A}(\mathbf{x})} \cdot \frac{\delta B}{\delta \mathbf{\Pi}(\mathbf{x})} \right] \quad (2.88)$$

Here, functional derivatives with respect to \mathbf{A} and $\mathbf{\Pi}$ appear for the fields. The Poisson bracket is antisymmetric in A and B and satisfies a ‘Jacobi identity’ (as does the commutator).

By working out $\{H, \mathbf{r}_\alpha\}$, only the second term of the first line in (2.88) contributes, and we get (after some calculations) the relativistic relation between velocity and momentum, Eq.(2.82). This is left as an exercise. Similarly, one gets $\dot{\mathbf{A}} = \mathbf{\Pi}/\varepsilon_0$.

A more complicated calculation is needed for $\dot{\mathbf{\Pi}}$ where we have to evaluate

$$-\frac{\delta}{\delta \mathbf{A}(\mathbf{x})} \sqrt{[\mathbf{p}_\alpha - e_\alpha \mathbf{A}(\mathbf{r}_\alpha)]^2 c^2 + m^2 c^4} \quad (2.89)$$

We note first that

$$\frac{\delta \mathbf{A}(\mathbf{r}_\alpha)}{\delta \mathbf{A}(\mathbf{x})} = \delta(\mathbf{r}_\alpha - \mathbf{x}) \quad (2.90)$$

because the ‘evaluation functional’ $\mathbf{A} \mapsto \mathbf{A}(\mathbf{r}_\alpha)$ is of course a linear functional. (For mathematicians, this property *defines* the δ -function.) To proceed, we use the usual rules of differential calculus and get

$$-\frac{\delta}{\delta \mathbf{A}(\mathbf{x})} \sqrt{[\mathbf{p}_\alpha - e_\alpha \mathbf{A}(\mathbf{r}_\alpha)]^2 c^2 + m^2 c^4} = \delta(\mathbf{r}_\alpha - \mathbf{x}) \frac{e_\alpha (\mathbf{p}_\alpha - e_\alpha \mathbf{A}(\mathbf{r}_\alpha)) c^2}{\sqrt{[\mathbf{p}_\alpha - e_\alpha \mathbf{A}(\mathbf{r}_\alpha)]^2 c^2 + m^2 c^4}} \quad (2.91)$$

Performing the same calculations as for the particles’ equations of motion, this can be written as the current density $\mathbf{j}(\mathbf{x})$.

The last term now involves the derivative

$$-\frac{\delta}{\delta \mathbf{A}(\mathbf{x})} \frac{1}{2\mu_0} \int d^3x (\nabla \times \mathbf{A})^2 \quad (2.92)$$

that we handle with the mathematical definition (2.84). Consider a small change $\mathbf{a}(\mathbf{x})$ of the vector potential. To linear order, this gives a change

$$\begin{aligned} (\nabla \times (\mathbf{A} + \mathbf{a}))^2 - (\nabla \times \mathbf{A})^2 &\approx 2(\nabla \times \mathbf{a}) \cdot (\nabla \times \mathbf{A}) \\ &= 2\nabla \cdot [\mathbf{a} \times (\nabla \times \mathbf{A})] + 2\mathbf{a} \cdot [\nabla \times (\nabla \times \mathbf{A})] \end{aligned} \quad (2.93)$$

the first term is a divergence and vanishes after integrating over all space. The second one contains a multiplied with a weighting function so that we get

$$-\frac{1}{2\mu_0} \frac{\delta}{\delta \mathbf{A}(\mathbf{x})} \int d^3x (\nabla \times \mathbf{A})^2 = -\frac{1}{\mu_0} \nabla \times (\nabla \times \mathbf{A}) \quad (2.94)$$

Putting everything together, we have for the equation of motion of the vector potential:

$$\dot{\mathbf{\Pi}} = \varepsilon_0 \ddot{\mathbf{A}} = \mathbf{j}(\mathbf{x}) - \frac{1}{\mu_0} \nabla \times (\nabla \times \mathbf{A}) \quad \dots \text{wrong} \quad (2.95)$$

which is nearly equivalent to the wave equation (2.7). The point is that the source term is the ‘full current’, not its transverse part. This is actually an error in our calculation because we did not take into account the fact that the vector potential is restricted to be transverse.

A simple way to repair this is to use \mathbf{A}_\perp in the Lagrangian (2.80) and the Hamiltonian (2.86). Now, the link between \mathbf{A}_\perp and the ‘full’ \mathbf{A} is a linear functional (actually, a linear projector).

This can be seen in the following way: consider an arbitrary \mathbf{A} and perform a gauge transformation (2.75) to remove the nonzero divergence. This fixes the ‘gauge function’ to satisfy

$$\nabla^2 \chi = -\nabla \cdot \mathbf{A} \quad (2.96)$$

whose solution (vanishing at infinity) is given by a ‘superposition of Coulomb potentials’:

$$\chi(\mathbf{x}) = \frac{1}{4\pi} \int d^3 x' \frac{\nabla' \cdot \mathbf{A}(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \quad (2.97)$$

where ∇' means the gradient with respect to \mathbf{x}' . After the gauge transformation, the now transverse vector potential is given by

$$\mathbf{A}_\perp(\mathbf{x}) = \mathbf{A}(\mathbf{x}) + \frac{1}{4\pi} \int d^3 x' [\nabla' \cdot \mathbf{A}(\mathbf{x}')] \nabla \frac{1}{|\mathbf{x} - \mathbf{x}'|} \quad (2.98)$$

Now, $\nabla(1/r) = -\nabla'(1/r)$, and after one integration by parts we have

$$A_{\perp i}(\mathbf{x}) = \int d^3 x' \delta_{ij}^\perp(\mathbf{x} - \mathbf{x}') A_j(\mathbf{x}') \quad (2.99)$$

where the transverse δ -function appears for which we have just derived the following representation

$$\delta_{ij}^\perp(\mathbf{x} - \mathbf{x}') = \delta_{ij} \delta(\mathbf{x} - \mathbf{x}') + \frac{1}{4\pi} \partial_j' \partial_i' \frac{1}{|\mathbf{x} - \mathbf{x}'|} \quad (2.100)$$

A careful evaluation of the second derivatives yields the explicit result

$$\delta_{ij}^\perp(\mathbf{x} - \mathbf{x}') = \frac{2}{3} \delta_{ij} \delta(\mathbf{x} - \mathbf{x}') - \frac{1}{4\pi} \left(\frac{\delta_{ij}}{r^3} - 3 \frac{r_i r_j}{r^5} \right), \quad \mathbf{r} = \mathbf{x} - \mathbf{x}' \quad (2.101)$$

By construction, the transverse δ -function acts like a usual δ -function on vector fields that are already transverse. We can interpret it as the ‘unit operator’ in the space of transverse vector fields.

Finally, if we write \mathbf{A}_\perp in the Hamiltonian, the equations of motion for $A_{\perp i}(\mathbf{x})$ lead to the following term

$$\int d^3 x' \frac{\delta H}{\delta \Pi_j(\mathbf{x}')} \frac{\delta A_{\perp i}(\mathbf{x})}{\delta A_j(\mathbf{x}')} = \int d^3 x' \frac{\delta H}{\delta \Pi_j(\mathbf{x}')} \delta_{ij}^\perp(\mathbf{x} - \mathbf{x}') = \left(\frac{\delta H}{\delta \Pi(\mathbf{x}')} \right)_{\perp i} \quad (2.102)$$

The functional derivative with respect to \mathbf{A} makes the transverse δ -function appear. The integral over \mathbf{x}' then projects the functional derivative with respect to Π into the transverse subspace. In this way, the time derivative of \mathbf{A}_\perp is transverse, as it should be in order to maintain the Coulomb gauge at all times.

The same procedure applies to the equation of motion for the field momentum: the bottom-line is that the source current \mathbf{j} is ‘transversalized’, leading to the correct wave equation (2.7).

2.4.4 Quantization

Mode expansion. The next step is to look for ‘normal modes’ of this field theory. We first identify some general requirements for the modes. The passage to plane waves is a little bit

tricky and is treated in detail in Sec. 2.1.2. Note that we ignore for the moment the matter-field coupling: we focus on the field Hamiltonian only.

We adopt the expansion

$$\begin{pmatrix} \mathbf{A}(\mathbf{x}, t) \\ \mathbf{\Pi}(\mathbf{x}, t) \end{pmatrix} = \sum_{\kappa} \mathbf{f}_{\kappa}(\mathbf{x}) \begin{pmatrix} q_{\kappa}(t) \\ p_{\kappa}(t) \end{pmatrix} \quad (2.103)$$

where the ‘mode functions’ $\mathbf{f}_{\kappa}(\mathbf{x})$ carry the space dependence and the ‘coordinates’ $q_{\kappa}(t)$ and ‘momenta’ $p_{\kappa}(t)$ the time-dependence.

The Coulomb gauge requires, of course, $\nabla \cdot \mathbf{f}_{\kappa}(\mathbf{x}) = 0$: the mode functions must be transverse.

We want the Hamiltonian to adopt a simple form using these modes. The space integral over $\mathbf{\Pi}^2$ becomes simple if we impose the modes to be orthogonal:

$$\int d^3x \mathbf{f}_{\kappa}(\mathbf{x}) \cdot \mathbf{f}_{\kappa'}(\mathbf{x}) = N_{\kappa} \delta_{\kappa\kappa'} \quad (2.104)$$

where N_{κ} is a normalization constant that we fix later. The momentum-part of the Hamiltonian then becomes

$$\frac{1}{2\varepsilon_0} \int d^3x \mathbf{\Pi}^2 = \sum_{\kappa} N_{\kappa} \frac{p_{\kappa}^2}{2\varepsilon_0} \quad (2.105)$$

Similarly, for the integral over $(\nabla \times \mathbf{A})^2$. Integrating by parts:

$$\begin{aligned} & \int d^3x (\nabla \times \mathbf{f}_{\kappa}) \cdot (\nabla \times \mathbf{f}_{\kappa'}) \\ &= \int d\mathbf{A} \cdot [\mathbf{f}_{\kappa} \times (\nabla \times \mathbf{f}_{\kappa'})] + \int d^3x \mathbf{f}_{\kappa} \cdot [\nabla \times (\nabla \times \mathbf{f}_{\kappa'})] \end{aligned} \quad (2.106)$$

The volume integral, strictly speaking, must be evaluated over a finite volume only, otherwise, we could not work with a discrete set of mode labels κ . The boundary term can nevertheless be made to vanish if either (i) the mode function \mathbf{f}_{κ} or its curl $(\nabla \times \mathbf{f}_{\kappa'})$ is required to vanish on the boundary of the volume or (ii) periodic boundary conditions on ‘opposite faces’ of a cubic volume are assumed. The case (i) is appropriate for modes in a cavity with perfectly conducting boundaries: then, \mathbf{f}_{κ} is proportional to the electric field, and the integrand in Eq.(2.106) vanishes because the field is normal to the boundary. The case (ii) is the favorite one for theorists because the mode functions can be taken as plane waves. Note that the eigenfrequencies of the two cavities are not the same. We ignore for the moment the complications of complex mode functions (see details below and the exercises) and continue.

The volume integral in (2.106) is reduced to the orthogonality relation if we require the mode functions to be eigenfunctions of the (vector) Helmholtz equation:

$$\nabla \times (\nabla \times \mathbf{f}_{\kappa}) = \varepsilon_0 \mu_0 \omega_{\kappa}^2 \mathbf{f}_{\kappa} \quad (2.107)$$

(This equation is actually equivalent to the scalar Helmholtz equation for all components of \mathbf{f}_{κ} because of transversality.)

With all these assumptions taken together, the Hamiltonian for the field takes the form

$$H_F = \sum_{\kappa} \left[\frac{N_{\kappa}}{2\varepsilon_0} p_{\kappa}^2 + \frac{\varepsilon_0}{2} N_{\kappa} \omega_{\kappa}^2 q_{\kappa}^2 \right] \quad (2.108)$$

We now fix the normalization to be $N_\kappa = 1$ and get a sum of harmonic oscillator Hamiltonians, one for each mode κ with ‘mass’ ε_0 and frequency ω_κ .

Note: the construction of field modes is a ‘classical problem’ of electrodynamics, it has nothing to do with quantum mechanics. The word ‘quantization volume’ that is sometimes used (to ensure that the mode index κ is discrete) is therefore misleading. Quantization is something different, as we shall see now.

Mode operators. Quantization proceeds by promoting the p_κ and q_κ to operators with the commutation relations

$$\frac{i}{\hbar} [p_\kappa, q_{\kappa'}] = \delta_{\kappa\kappa'} \quad (2.109)$$

The choice for this commutator is similar to the one for the particle coordinates and momenta in ordinary quantum mechanics. So in the end, field quantization is nothing else but ordinary quantization, once the dynamics of the field is reduced to a discrete set of ‘normal modes’. The procedure that we have followed was first laid out by Dirac. It is called ‘canonical quantization’.

The commutator between the fields becomes

$$\frac{i}{\hbar} [\Pi_i(\mathbf{x}), A_j(\mathbf{x}')] = \sum_\kappa f_{\kappa i}(\mathbf{x}) f_{\kappa j}(\mathbf{x}') \stackrel{!}{=} \delta_{ij}^\perp(\mathbf{x} - \mathbf{x}') \quad (2.110)$$

The last equality is obtained by applying the canonical quantization scheme directly to the fields, care being taken that the fields (operators) ‘live’ in the space of transverse vector fields (operators). It means that the mode functions $\mathbf{f}(\mathbf{x})$ form a *complete set* of functions in the transverse field space. To implement this equality, one takes in practice the limit of an infinitely large quantization volume where the sum degenerates into an integral. With discrete functions, one can actually represent only a ‘finite volume version’ of the transverse δ -function—the one that comes by applying the finite volume boundary conditions to the equation (2.96).

For the harmonic oscillator, creation and annihilation operators are a convenient tool to construct the Hilbert space of quantum states. In our context, these operators, a_κ^\dagger and a_κ , correspond to the ‘creation’ and ‘destruction’ of one ‘photon’. The mode coordinate and momentum operators are given by

$$q_\kappa = \sqrt{\frac{\hbar}{2\varepsilon_0\omega_\kappa}} (a_\kappa + a_\kappa^\dagger) \quad (2.111)$$

$$p_\kappa = \sqrt{\frac{\hbar\varepsilon_0\omega_\kappa}{2}} i (a_\kappa^\dagger - a_\kappa) \quad (2.112)$$

where we continue to write ε_0 for the oscillator mass and where the commutation relation is

$$[a_\kappa, a_{\kappa'}^\dagger] = \delta_{\kappa\kappa'} \quad (2.113)$$

The field Hamiltonian then takes the form

$$H_F = \sum_\kappa \frac{\hbar\omega_\kappa}{2} (a_\kappa a_\kappa^\dagger + a_\kappa^\dagger a_\kappa) = \sum_\kappa \hbar\omega_\kappa (a_\kappa^\dagger a_\kappa + \frac{1}{2}) \quad (2.114)$$

The last way of writing makes two essential things explicit:

- the energies (the energy eigenvalues!) of a given field mode are quantized in units of $\hbar\omega_\kappa$ (a ‘photon energy’) and the ‘number of photons’ is represented by the operator $a_\kappa^\dagger a_\kappa$.
- The ground state of the field corresponds to the state $|\text{vac}\rangle$ such that $a_\kappa|\text{vac}\rangle = 0$ for all κ . This is an energy eigenstate whose energy is infinite, $\frac{1}{2} \sum_\kappa \hbar\omega_\kappa$, the sum over the ‘zero-point energies’ of all the modes.

To summarize, we give the mode expansions of the vector potential that we have found

$$\mathbf{A}(\mathbf{x}, t) = \sum_\kappa \sqrt{\frac{\hbar}{2\varepsilon_0\omega_\kappa}} \mathbf{f}_\kappa(\mathbf{x}) (a_\kappa e^{-i\omega_\kappa t} + a_\kappa^\dagger e^{i\omega_\kappa t}). \quad (2.115)$$

We have used here the Heisenberg picture for the vector potential operator. From the Hamiltonian (2.114), it is easy to show that the operator $a_\kappa(t)$ evolves with a complex exponential $e^{-i\omega_\kappa t}$. This is also called a ‘positive frequency operator’. It can be shown that operators that destroy particles and lower the energy of a quantum state are always positive frequency operators.

Plane wave modes. For completeness, we give here the plane-wave expansion for the field mode functions. These are complex, and therefore they are normalized according to

$$\int d^3x \mathbf{f}_\kappa^*(\mathbf{x}) \cdot \mathbf{f}_{\kappa'}(\mathbf{x}) = \delta_{\kappa\kappa'} \quad (2.116)$$

instead of Eq.(2.104). This can be ensured with the choice

$$\mathbf{f}_\kappa(\mathbf{x}) = \frac{1}{\sqrt{V}} \mathbf{e}_\kappa e^{i\mathbf{k}\cdot\mathbf{x}} \quad (2.117)$$

where V is the volume of the box with periodic boundary conditions, \mathbf{k} is a discrete wave vector: it increases in steps of $2\pi/V^{1/3}$ for a cubic box. And \mathbf{e}_κ is a ‘transverse’ polarization vector with the property $\mathbf{k} \cdot \mathbf{e}_\kappa = 0$. There are two mutually orthogonal choices of polarization for a given \mathbf{k} . (These can be complex, describing circular polarization.) The frequency of this mode is $\omega_\kappa = |\mathbf{k}|/\sqrt{\varepsilon_0\mu_0} = c|\mathbf{k}|$, as can be seen from the Helmholtz equation (2.107). The magnetic field is oriented along $\mathbf{k} \times \mathbf{e}_\kappa \equiv (\omega_\kappa/c)\mathbf{b}_\kappa$.

Finally, the quantized vector potential, electric and magnetic fields are given in terms of the following plane wave expansion

$$\mathbf{A}_\perp(\mathbf{x}, t) = \sum_\kappa \sqrt{\frac{\hbar}{2\varepsilon_0\omega_\kappa V}} \left(\mathbf{e}_\kappa e^{i(\mathbf{k}\cdot\mathbf{x} - \omega_\kappa t)} a_\kappa + \text{h.c.} \right). \quad (2.118)$$

$$\mathbf{E}_\perp(\mathbf{x}, t) = \sum_\kappa \sqrt{\frac{\hbar\omega_\kappa}{2\varepsilon_0 V}} \left(i\mathbf{e}_\kappa e^{i(\mathbf{k}\cdot\mathbf{x} - \omega_\kappa t)} a_\kappa + \text{h.c.} \right). \quad (2.119)$$

$$\mathbf{B}(\mathbf{x}, t) = \sum_\kappa \sqrt{\frac{\hbar\omega_\kappa\mu_0}{2V}} \left(i\mathbf{b}_\kappa e^{i(\mathbf{k}\cdot\mathbf{x} - \omega_\kappa t)} a_\kappa + \text{h.c.} \right). \quad (2.120)$$

Sometimes, you may encounter these formulas without the factor i . Then the operators $ia_{\mathbf{k}\lambda}$ are being used instead, but they have the same commutation relations. Note that Eq.(2.119) gives only the ‘transverse’ part of the electric field. The ‘longitudinal’ part, $-\nabla\phi$, is determined according to (2.5) by the charge density.

Note: A useful shortcut to derive the prefactors is the following: for each mode, match the energy density $\frac{1}{2}\epsilon_0\mathbf{E}^2 + (1/2\mu_0)\mathbf{B}^2$ to the photon energy per quantization volume, $(\hbar\omega_{\kappa}/V)(a_{\kappa}^{\dagger}a_{\kappa} + \frac{1}{2})$. In free space, the electric and magnetic energy densities are equal. Average over the spatial oscillations in \mathbf{E}^2 for simplicity.

Exercise. Show that the momentum of the field can be written as a sum over modes as well. With plane wave modes, the momentum per mode is quantized in units of $\hbar\mathbf{k}$, as expected. Arbitrary cavity modes involving sin or cos functions are not eigenfunctions of the momentum operators, therefore their momentum is not well-defined. A ‘cavity photon’ therefore does not have a well-defined momentum.

Exercise. Write the equation of motion for a mode operator a_{κ} and include the source current. Solve it for known time-dependence of current.

Chapter 3

Evidence for vacuum fluctuations

Electromagnetic energy density in the vacuum state, see Sec.2.3.3.

Lamb shift, see p80 in the book by Migdal.

Spontaneous decay of excited state.

Casimir energy (difference of zero-point energies), see Sec.3.2.

3.1 Dirac's calculation of spontaneous decay

Initial state of atom and field: $|\psi(0)\rangle = |e\rangle \otimes |\text{vac}\rangle = |e, \text{vac}\rangle$. Ansatz for later times:

$$|\psi(t)\rangle = c_e(t)|e, \text{vac}\rangle + \sum_{\kappa} c_{\kappa}(t)|g, 1_{\kappa}\rangle \quad (3.1)$$

where $\kappa = (\mathbf{k}, \mu)$ is a shorthand for the momentum and polarization quantum numbers of one-photon states. We shall solve the time-dependent Schrödinger equation with perturbation theory (see Sec.1.3.2) The result is that the excited state decays spontaneously and in an approximately exponential manner: the probability p_e of finding an atom in its excited state follows the *rate equation*

$$\frac{dp_e}{dt} = -\gamma_e p_e, \quad \gamma_e = \frac{|\mathbf{d}|^2 \omega_A^3}{3\pi\epsilon_0 \hbar c^3} \quad (3.2)$$

where $\mathbf{d} = \langle g|\hat{\mathbf{d}}|e\rangle$ is the matrix element of the electric dipole operator (the “transition dipole”) and ω_A the Bohr frequency.

Outline of the calculation. The initial condition in the atom+field state (3.1) is $|\psi(0)\rangle = |e, \text{vac}\rangle$. Equation of motion for the amplitudes in the interaction picture

$$i\hbar\partial_t c_e(t) = \langle e, \text{vac}|H_{\text{int}}(t)|\psi(t)\rangle \quad (3.3)$$

involves interaction Hamiltonian

$$H_{\text{int}} = -\mathbf{d} \cdot \mathbf{E}(\mathbf{r}_A) = -\left(\mathbf{d}_{ge}\sigma + \mathbf{d}_{ge}^*\sigma^\dagger\right) \sum_{\kappa} \mathcal{E}(\omega_{\kappa}) \left(\mathbf{e}_{\kappa} e^{i\mathbf{k}\cdot\mathbf{r}_A} a_{\kappa} + \text{h.c.}\right) \quad (3.4)$$

with the time dependence $H_{\text{int}}(t)$ given by the free evolution of the atom and field operators. We require the matrix element

$$\begin{aligned} \langle e, \text{vac} | H_{\text{int}}(t) | g, 1_{\kappa} \rangle &= -\langle e, \text{vac} | \mathbf{d}_{ge}^* \sigma^\dagger e^{i\omega_A t} \sum_{\kappa'} \mathcal{E}(\omega_{\kappa'}) \mathbf{e}_{\kappa'} e^{i\mathbf{k}'\cdot\mathbf{r}_A} a_{\kappa'} e^{-i\omega_{\kappa'} t} | g, 1_{\kappa} \rangle \\ &= -\mathbf{d}_{ge}^* \cdot \mathbf{e}_{\kappa} \mathcal{E}(\omega_{\kappa}) e^{i\mathbf{k}\cdot\mathbf{r}_A} e^{i(\omega_A - \omega_{\kappa})t} =: \hbar g_{\kappa} e^{i(\omega_A - \omega_{\kappa})t} \end{aligned} \quad (3.5)$$

The other operators do not contribute to this matrix element. A similar calculation gives the equation of motion

$$i\partial_t c_{\kappa}(t) = g_{\kappa}^* e^{-i(\omega_A - \omega_{\kappa})t} c_e(t) \quad (3.6)$$

To proceed, we use time-dependent perturbation theory, similar to Sec.1.3.2. To lowest order, the amplitudes in the interaction picture are constant:

$$c_e(t) = c_e^{(0)}(t) + c_e^{(1)}(t) + \dots = c_e^{(0)}(0) + c_e^{(1)}(t) + \dots \quad (3.7)$$

We can insert the initial value $c_e^{(0)}(0) = 1$ and integrate Eq.(3.6) to get at first order

$$\begin{aligned} c_{\kappa}(t) &= \underbrace{c_{\kappa}^{(0)}(t)}_0 + c_{\kappa}^{(1)}(t) + \dots \\ &= \frac{g_{\kappa}^*}{\omega_A - \omega_{\kappa}} \left(e^{-i(\omega_A - \omega_{\kappa})t} - 1 \right) \\ &= i g_{\kappa}^* e^{-i(\omega_A - \omega_{\kappa})t/2} \frac{\sin(\omega_A - \omega_{\kappa})t/2}{(\omega_A - \omega_{\kappa})/2} \end{aligned} \quad (3.8)$$

We now define the spontaneous decay rate by

$$\begin{aligned} \gamma_e &= \frac{\text{prob of finding atom in } |g\rangle}{t} \\ &= \frac{1}{t} \sum_{\kappa} |c_{\kappa}(t)|^2 \end{aligned} \quad (3.9)$$

The probabilities are summed over the modes, using the fact that these are distinguishable states of the field. Insert the expression for the coupling constants g_{κ} and integrate over all modes ($\kappa = (\mathbf{k}, \lambda)$)

$$\frac{1}{t} \sum_{\kappa} |c_{\kappa}(t)|^2 = \frac{V}{t} \int \frac{d^3k}{(2\pi)^3} \sum_{\lambda} |\mathbf{d}_{ge}^* \cdot \mathbf{e}_{\mathbf{k}\lambda}|^2 |\mathcal{E}(\omega_{\kappa})|^2 \frac{\sin^2(\omega_A - \omega_{\kappa})t/2}{(\omega_A - \omega_{\kappa})^2/4}$$

Polarization sum

$$\sum_{\lambda} |\mathbf{d}_{ge}^* \cdot \mathbf{e}_{\mathbf{k}\lambda}|^2 = |\mathbf{d}_{ge}^*|^2 - |\mathbf{d}_{ge}^* \cdot \hat{\mathbf{k}}|^2 \quad (3.10)$$

Angular integral

$$\int d\Omega(\hat{\mathbf{k}}) |\mathbf{d}_{ge}^* \cdot \hat{\mathbf{k}}|^2 = \frac{4\pi}{3} |\mathbf{d}_{ge}^*|^2 \quad (3.11)$$

gives (*Faktor 2 fehlt in der Vorlesung*)

$$\begin{aligned} \gamma_e &= \frac{8\pi V}{3t} |\mathbf{d}_{ge}^*|^2 \int_0^{\infty} k^2 dk \frac{\hbar ck}{(2\pi)^3 2\varepsilon_0 V \hbar^2} \frac{\sin^2(\omega_A - ck)t/2}{(\omega_A - ck)^2/4} \\ &= \frac{8\pi |\mathbf{d}_{ge}^*|^2}{3(2\pi)^3 t} \int_0^{\infty} \frac{\omega^2 d\omega}{c^3} \frac{\omega}{2\varepsilon_0 \hbar} \frac{\sin^2(\omega_A - \omega)t/2}{(\omega_A - \omega)^2/4} \end{aligned} \quad (3.12)$$

Limit of short times: expand the sin and get $\gamma \sim t$. The coefficient involves a divergent integral over ω^3 , however. This must be cut off at high frequencies where the atom-field coupling in this form is no longer valid. This regime may be called the “quantum Zeno effect”: by (not) observing the atom in the ground state, the excited state “freezes” and does not decay. In practise, this measurement requires such short times that it cannot be done.

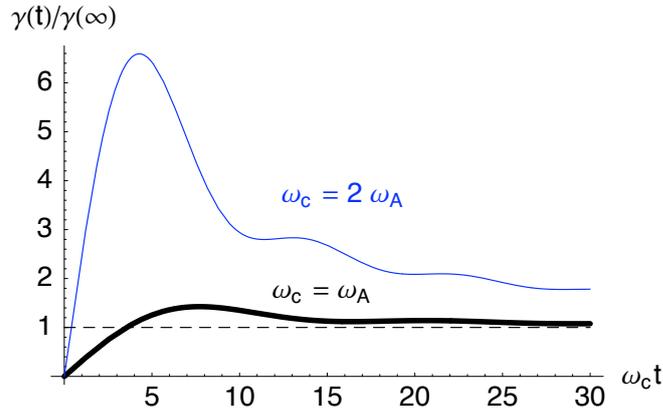


Figure 3.1: Time-averaged decay rate $\gamma(t)$ for spontaneous decay, defined in Eq.(3.9) and calculated from Eq.(3.12). Here: sharp cutoff frequency ω_c . The Zeno effect corresponds to short times where $\gamma(t) \rightarrow 0$. At large times, convergence to a constant decay rate $\gamma(\infty) = \gamma_e$.

Limit of larger times: not too large because the perturbation theory breaks down. One uses the approximation (*Faktor 2 fehlt in der Vorlesung*)

$$\frac{\sin^2(\omega_A - \omega)t/2}{(\omega_A - \omega)^2/4} \approx 2\pi|t| \delta(\omega_A - \omega) \quad (3.13)$$

Note that this removes the factor $1/t$ to get from Eq.(3.12) the constant decay rate

$$\gamma_e \approx 2\pi \cdot 4\pi |\mathbf{d}_{ge}^*|^2 \frac{2}{3(2\pi)^3} \frac{\omega^3}{2\varepsilon_0 \hbar c^3} = \frac{|\langle e|\mathbf{d}|g\rangle|^2 \omega_A^3}{3\pi\varepsilon_0 \hbar c^3} \quad (3.14)$$

which is the result found by Dirac in 1927. The time-dependent decay rate $\gamma_e(t)$ with a sharp cutoff at ω_c in the frequency integral is illustrated in Fig.3.1. One sees that the constant value $\gamma_e(\infty)$ [given by Eq.(3.50)] is reached on a time scale given by roughly $1/\omega_c$ where the ω_c is the cutoff frequency. At large cutoff, the deviation from the constant is larger: there is in fact a contribution $\sim \omega_c^2/t$.

The transient time over which the decay rate rises to its stationary value can be identified with the *correlation time* of vacuum fluctuations – in so far these couple to the atom. This correlation time is *short* (of the order of the inverse frequency cutoff $1/\omega_c$ or the optical period $1/\omega_A$), meaning that the vacuum fluctuations behave approximately as “white (quantum) noise”. The noise is quantum because the ground state remains stable (at zero temperature), only the excited state decays.

There is a relation between the correlation time (defined from the autocorrelation function $\langle E_i(\mathbf{x}, t + \tau)E_j(\mathbf{x}, t) \rangle$) and the spectrum of vacuum fluctuations (this is called the Wiener-Khintchine theorem). The correlation time is defined as the time scale τ_c on which the autocorrelation decays to zero (it factorizes, but we consider here a state where the averages $\langle E_j(\mathbf{x}, t) \rangle$ are zero). The spectrum is the Fourier transform of the autocorrelation (there are different conventions for the sign of the exponential here)

$$S_{ij}(\mathbf{x}; \omega) = \int d\tau e^{-i\omega\tau} \langle E_i(\mathbf{x}, t + \tau)E_j(\mathbf{x}, t) \rangle \quad (3.15)$$

Now there is an “uncertainty relation” between the frequency width $\Delta\omega$ of the spectrum $S_{ij}(\mathbf{x}; \omega)$ and the correlation time $\tau_c \sim 1/\Delta\omega$. In other words: if the field has a short correlation time, then its spectrum is broad band. In the limit of a flat spectrum, one speaks of *white noise*, and the correlation time becomes arbitrarily short. The “quantum noise” one encounters in quantum optics is nearly white. Many measurements and processes happen on time scales larger than τ_c where spontaneous emission happens at a constant rate.

Short cut. The spontaneous decay rate can also be found in a semiclassical picture: if one models the excited atom by a classical dipole with complex amplitude \mathbf{d}_{ge} and frequency ω_A , $\mathbf{d}_{sc}(t) = \mathbf{d}_{ge} e^{-i\omega_A t}$, then the power radiated by this dipole (a classical result of electrodynamics) can be written as the product

$$P_{\text{rad}} = \frac{|\mathbf{d}_{ge}|^2 \omega_A^4}{3\pi\varepsilon_0 c^3} = \hbar\omega_A \gamma_e \quad (3.16)$$

the decay rate thus corresponds, in this picture, as the time for radiating an energy equal to one photon energy. This shortcut “explains” why the decay rate γ_e involves the factor $1/\hbar$. A more detailed discussion is suggested in the exercises. The picture does not capture the fluctuations of the radiation field emitted by the atom. It is also easy to see that the average value of the field (operator) actually vanishes in the state $|\psi(t)\rangle$ of Eq.(3.1), although the average Poynting vector is nonzero.

Refinements. This calculation can be refined in several ways.

Our definition of the decay rate $\gamma_e(t)$ is a “time average” (we divide by the observation time t in Eq.(3.9)). One could equally well compute the derivative $d/dt \sum_{\kappa} |c_{\kappa}(t)|^2$ which gives a slightly different picture.

The distinction between “short” and “long” times cannot be avoided. At short times, the evolution can be corrected by taking into account non-resonant parts in the atom+field state, i.e. including terms $|e, 2_{\kappa}\rangle$ and, of course, atomic states other than the two levels $|g\rangle, |e\rangle$. All these states appear beyond the resonance approximation.

A frequency cutoff is systematically used to regularize the divergent integrals. (Other techniques like “dimensional” or “ ζ -function regularization are also available.) This can be justified on physical grounds, since the atom will not react to high-frequency fields, the fields are not constant across the atom’s size etc. In condensed-matter theory, cutoff frequencies are related to the nature of the environment (phonons in a solid instead of free-space photons) and estimates are available.

The perturbation theory is only valid for short enough time: clearly, as soon as the probability $\gamma_e t \sim 1$, it must break down. Calculations that go beyond are based, e.g., on the resolvent operator for the atom+field Hamiltonian. The unstable state $|e, \text{vac}\rangle$ appears in this context as a *resonance*, i.e., a pole of the resolvent operator at a complex-valued energy

$$E_e \approx \hbar (\omega_A + \delta\omega_A) - i\hbar\gamma_e/2 \quad (3.17)$$

whose real part contains the Lamb-shift $\delta\omega_A$ and whose imaginary part the decay rate. With this “pole approximation”, the time-dependent Schrödinger equation is solved with a damped exponential

$$c_e(t) \approx c_e(0) e^{-\gamma_e t} e^{-i\delta\omega_A t} \quad (3.18)$$

which is valid even at long times ($\gamma_e t \gg 1$). Technically, one can compute the resolvent by inserting the solution for $c_\kappa(t)$ [Eq.(3.8)] into the equation for $c_e(t)$ [Eq.(3.3)]. The result is an integro-differential equation of motion that can be solved with a Laplace transformation. The complex energy (3.17) is a pole of the Laplace transform of $c_e(t)$. In general, one also finds other poles and/or branch cuts. These lead to a non-exponential decay, either by adding extra decay constants or even an algebraic term (a power law in $1/t$).

3.2 Casimir energy

The Casimir force is the attraction between two metallic mirrors placed in vacuum. It is interpreted in terms of the change in the zero-point energy (the famous $\frac{1}{2}\hbar\omega$ of the harmonic oscillator ground state) induced by the presence of the mirrors. We give here an sketch of the calculation done by Casimir around 1948 [Proc. Kon. Ned. Akad. Wet. 51 (1948) 793].

We consider the ground state energy of the multi-mode electromagnetic field

$$E_0 = \sum_{\mathbf{k}\lambda} \frac{\hbar\omega_{\mathbf{k}\lambda}}{2}$$

that is of course infinite and compare the cases of a planar cavity formed by two mirrors (distance L) and empty space (i.e., two mirrors infinitely apart). In the first case, we have standing wave modes between the mirrors with a frequency

$$\omega^{(cav)} = c\sqrt{K^2 + k_n^2}, \quad k_n = \frac{n\pi}{L}$$

with $K^2 = k_x^2 + k_y^2$ and $n = 1, 2, \dots$, while in empty space,

$$\omega = c\sqrt{K^2 + k_z^2}$$

with $-\infty < k_z < \infty$. We first compute the difference in the electromagnetic mode density per volume AL where A is the ‘quantization area’ in the xy -plane. We cheat with the polarizations and multiply by a factor 2:

$$\begin{aligned} \rho_L(\omega) &= \frac{4\pi}{AL} \sum_{\mathbf{K},n} \delta\left(\omega - c\sqrt{K^2 + k_n^2}\right) \\ &= \frac{2}{L} \int_0^\infty K dK \sum_n \delta\left(\omega - c\sqrt{K^2 + k_n^2}\right) \end{aligned} \quad (3.19)$$

The integration over K can be performed with the substitution $K \mapsto c\sqrt{K^2 + k_n^2}$ and gives

$$\rho_L(\omega) = \frac{2\omega}{Lc^2} \sum_n \Theta(\omega - ck_n) \quad (3.20)$$

where Θ is the step function. It arises because for a given n , there are no modes with frequency smaller than ck_n . The same calculation in the infinite volume gives

$$\rho_\infty(\omega) = \frac{2\omega}{c^2} \int_0^\infty \frac{dk_z}{\pi} \Theta(\omega - ck_z) \quad (3.21)$$

The k_z integral can of course be performed, but we keep it here to illustrate one of the basic features of the Casimir calculation: the result originates from the difference between a ‘discrete spectrum’ (the sum over the k_n) and a continuum (the integral over k_z).

The Casimir energy is now found as the difference in vacuum energy *per area* in the space of length L between the mirrors:

$$\Delta E = L \int_0^\Omega \frac{\hbar\omega}{2} (\rho_L(\omega) - \rho_\infty(\omega)) \quad (3.22)$$

We have introduced an upper cutoff frequency Ω because the integrals are likely to diverge in the UV. One of the mathematical difficulties (that we are not going to discuss here) is to what extent the results depend on the cutoff. At a suitable stage of the calculation, we are going to take the limit $\Omega \rightarrow \infty$, of course.

The ω -integrals can be performed before the sum over n (the integral over k_z), and one gets

$$\Delta E = \frac{\hbar}{6\pi c^2} \left[\sum_{n=1}^{\lfloor \Omega L/\pi c \rfloor} (\Omega^3 - (ck_n)^3) - \int_0^{\Omega/c} \frac{dk_z}{\pi} (\Omega^3 - (ck_z)^3) \right] \quad (3.23)$$

where $\lfloor x \rfloor$ is the largest integer smaller than x . Introducing the number $N = \Omega L/\pi c$ and the dimensionless integration variable $z = k_z L/\pi$, this can be written in the form

$$\Delta E = \frac{\hbar c \pi^2}{6L^3} \left[\sum_{n=1}^{\lfloor N \rfloor} (N^3 - n^3) - \int_0^N dz (N^3 - z^3) \right] \quad (3.24)$$

The difference in brackets is some magic number and equal to $-1/120$ in the limit $N \rightarrow \infty$. (This is an application of the Euler-MacLaurin formula for sums. An alternative proof is sketched below.) The Casimir energy of two mirrors is thus equal to

$$\Delta E = -\frac{\hbar c \pi^2}{720L^3} \quad (3.25)$$

so that the force per unit area is $F_C/A = -\hbar c\pi^2/240L^4$: since the energy decreases as $L \rightarrow 0$, the two mirrors placed in vacuum attract each other.

Note that this result is independent of the nature of the mirrors, as well as their electric charge. The electromagnetic field only enters inasmuch as its modes give a contribution to the energy of the vacuum state. Field theorists have computed the contribution to the Casimir energy from the Dirac electron field, for example. It is small if the mirror separation is large compared to the Compton wavelength $\hbar/mc \approx 2.5$ pm — which is nearly always the case. The Casimir energy, being attractive, is sometimes thought of a means to ‘stabilize’ a classical model of the electron (a bag of charge) against the Coulomb repulsion.

Sum minus integral

We use a trick in the complex plane. There is a theorem for functions f and D that are analytical in a domain limited by the integration contour \mathcal{C} :

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} dz f(z) \frac{d}{dz} \log D(z) = \sum_n f(z_n) \quad (3.26)$$

where the z_n are the zeros of D in the interior of the contour. We will use $f(z) = N^3 - z^3$ and choose $D(z)$ such that it is zero for the values $z_n = n$: $D(z) = \sin(\pi z)$. The differentiation under the integral sign gives

$$\sum_{n=0}^N (N^3 - n^3) = \frac{1}{2} \oint_{\mathcal{C}_N} dz (N^3 - z^3) \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \quad (3.27)$$

We chose an integration contour as shown in fig. 3.2 running from $+N$ above the real axis to 0 and going back to $+N$ below the real axis (the sum over all positive zeros of $\sin \pi z$ thus gives the sum on the left hand side). We are eventually interested in the limit $N \rightarrow \infty$. Make the following transformations on the upper and lower part of the contour:

$$\begin{aligned} \text{upper part:} \quad & \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} = -1 + \frac{2e^{i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \\ \text{lower part:} \quad & \frac{e^{i\pi z} + e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} = 1 + \frac{2e^{-i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \end{aligned}$$

The constants ± 1 give for both the upper and lower path an integral over $N^3 - z^3$ that can be combined into

$$\sum_{n=0}^N (N^3 - n^3) = \int_0^N dz (N^3 - z^3) + \oint_{\mathcal{C}} dz (N^3 - z^3) \frac{e^{\pm i\pi z}}{e^{i\pi z} - e^{-i\pi z}}$$

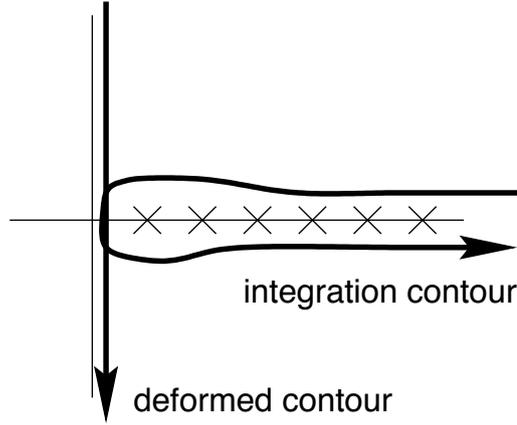


Figure 3.2: Integration contour for (3.27).

In the second integral, the exponential takes the appropriate sign on the upper and lower parts of the contour. The first integral on the right hand side is exactly the integral that we have to subtract in Eq.(3.24). The upper and lower parts of the contour can now be shifted onto the (positive or negative) imaginary axis because the integrand has no singularities (these are located on the real axis only). The quarter-circle with radius $|z| = N$ contributes only a negligible amount because of the $e^{\pm i\pi z}$.

Choosing $z = \pm it$ on the imaginary axis, we get

$$\begin{aligned}
 & \oint_c dz (N^3 - z^3) \frac{e^{\pm i\pi z}}{e^{i\pi z} - e^{-i\pi z}} \\
 &= -i \int_0^\infty dt \frac{[N^3 - (it)^3] e^{-\pi t}}{e^{-\pi t} - e^{\pi t}} - i \int_0^\infty dt \frac{[N^3 - (-it)^3] e^{-\pi t}}{e^{\pi t} - e^{-\pi t}} \\
 &= -2 \int_0^\infty \frac{dt t^3}{e^{2\pi t} - 1}
 \end{aligned}$$

Note that the imaginary parts of the two integrals that involve N^3 cancel each other: we have finally eliminated the cutoff.

You have encountered the last integral in the context of blackbody radiation. Changing to the integration variable $t' = 2\pi t$, the integral gives $1/240$, so that we have in the end

$$\lim_{N \rightarrow \infty} \left(\sum_{n=0}^N (N^3 - n^3) - \int_0^N dz (N^3 - z^3) \right) = -\frac{2}{240} = -\frac{1}{120} \quad (3.28)$$

as announced in the text.

3.3 Quantum Langevin formulation

3.3.1 Idea

Many systems in quantum optics are “open” and follow equations of motion that go beyond the Schrödinger equation. This applies to systems for which the Hamiltonian is not completely known or systems in contact with an “environment”. The equations of motion can be formulated as so-called master equations (similar to the Schrödinger picture, dynamics for the system density operator) or as so-called Langevin equations (similar to the Heisenberg picture, dynamics of the system operators). These equations are the quantum analogue of kinetic theories that describe the dynamics of a system including different dissipative effects. They provide, among others, the approach of the system towards thermal equilibrium, but also non-equilibrium states that are driven by external forces, the typical example being a laser field.

In quantum optics, the “system” can be an atom, a collection of atoms, or a field mode in a cavity. The system is “open” because it interacts with the “environment” (rest of the world), that is modelled as a continuum of quantized field modes. An atom decays irreversibly by emitting a photon into previously empty vacuum modes. This phenomenon of “spontaneous emission” does not allow for a description in terms of a Hamiltonian. In fact, the state of the atom does not remain pure, and entropy increases because the photon can be emitted into any direction of space. A cavity mode decays because photons escape through the cavity mirrors, effectively becoming excitations of the “modes outside the cavity”. Alternatively, the field energy can be absorbed by the material making up the mirrors.

We focus first on the derivation of a master equation for a two-level atom. We shall find from the general theory two results: (1) the rate of spontaneous emission of an electronically excited state and (2) Glauber’s formula for the signal of a photodetectors in terms of normally ordered correlation functions of the electric field operator.

Time scales

The derivation builds on the following hierarchy of time scales:

$$\frac{2\pi}{\omega_A} \leq \tau_c \ll \frac{2\pi}{\Omega} \sim \frac{1}{\gamma} \quad (3.29)$$

where the shortest time scale is the optical period of the light field that is near-resonant with the atomic transition frequency ω_A . This is typically smaller than a few fs (femtoseconds) (in the visible range).

The “correlation time” τ_c of the electromagnetic field describes qualitatively the time interval over which the quantum fluctuations of the electromagnetic field (that couple

significantly to the atom) are “smooth”. Two electric field measurements are not significantly “similar” if they are taken at intervals larger than τ_c . We give an estimate below, it turns out to be a few times larger than the optical period.

The third time scale is the Rabi period that scales with the inverse of the Rabi frequency of a laser. This depends of course on the laser power (and the atomic transition dipole), but typical values are in the 1–10 ns (nanoseconds) range. Note that this is much longer than the optical period and the field correlation time.

The last time scale is the lifetime of the excited state of the atom, inversely proportional to the spontaneous decay rate γ . This is typically in the 1–10 ns range as well, depending on the atom.

We thus have a situation similar to “Brownian motion”: the atom is “shaken around” by the vacuum field fluctuations that vary much faster than the typical dynamics for the atomic state (Rabi oscillations, spontaneous decay). The master equation takes advantage of this separation of time scales to find an equation of motion that can be applied on “slow time scales”. As a result, one gets a single equation that describes both, the “Hamiltonian” effects of a laser (Rabi oscillations) and the “dissipative / friction” effects due to vacuum fluctuations.

Hamiltonian

Two-level atom, field, and electric dipole interaction:

$$H = \frac{\hbar\omega_A}{2}\sigma_3 + \sum_k \hbar\omega_k a_k^\dagger a_k + \sum_k \hbar \left(g_k^* a_k^\dagger \sigma + g_k \sigma^\dagger a_k \right) \quad (3.30)$$

neglect the zero-point energy of the field. Coupling constant $\hbar g_k = -\mathbf{d}_{eg} \cdot \mathbf{f}_k(x_A)(\hbar\omega_k/2\varepsilon_0)^{1/2}$ with the mode function $\mathbf{f}_k(x)$ evaluated at the position of the atom. Sometimes called ‘vacuum Rabi frequency’.

In the following, we also use the (re-scaled) electric field operator

$$E(t) = \sum_k g_k a_k(t) + \text{h.c.} \quad (3.31)$$

Field correlation time

To get an estimate of the field correlation time, we compute the two-time correlation (or coherence) function

$$C(\tau) = \langle E(t+\tau)E(t) \rangle \quad (3.32)$$

If we use the coupling constants g_k to write the mode expansion of the quantized field (this is equivalent to re-scaling the electric field so that it has units of frequency), we get

$$\langle E(t+\tau)E(t) \rangle = \sum_k |g_k|^2 \langle a_k(t+\tau)a_k^\dagger(t) \rangle$$

$$= \sum_k |g_k|^2 e^{-i\omega_k \tau} \quad (3.33)$$

An explicit calculation using the mode amplitudes in free space leads to an integral of the form

$$C(\tau) \sim \int_0^\infty d\omega \omega^3 e^{-i\omega \tau} \quad (3.34)$$

which looks like a third derivative of a δ -function in τ . Well, not exactly since the integrations starts at $\omega = 0$. Generalizing to finite temperature, the Fourier transform of the correlation function (3.34) becomes

$$C(\omega) \sim \omega^3 \bar{n}(\omega) = \frac{\omega^3}{\exp(\hbar\omega/k_B T) - 1} \quad (3.35)$$

The sign convention of the Fourier transform is chosen here such that $\omega > 0$ corresponds to photon numbers $a^\dagger a$ ('normal order'), while at $\omega < 0$, the anti-normal order aa^\dagger is picked. See Fig.3.3: the spectrum is proportional to $-\omega^3$ for negative frequencies, $-\omega \gg k_B T/\hbar$. Near zero frequency, it has a quadratic behaviour, at positive frequencies, a maximum near $\omega = 2k_B T/\hbar$ (Wien displacement law) and an exponential decay for $\omega \gg k_B T/\hbar$.

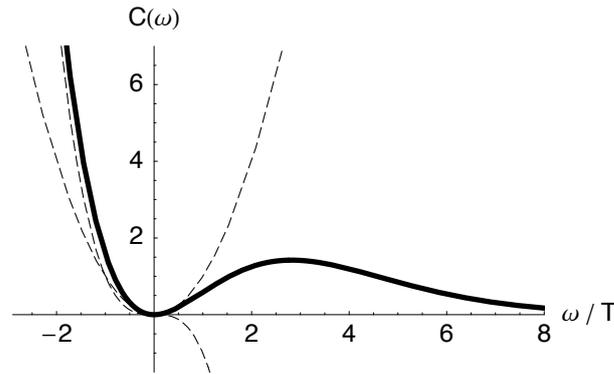


Figure 3.3: Spectrum $C(\omega)$, Eq.(3.35), of the radiation field (normal order) at finite temperature. The frequency is plotted in units of $k_B T/\hbar$. The dashed lines are ω^2 and $-\omega^3$.

Actually, we forgot one thing in this estimate: the Hamiltonian we start with is only valid for field modes k near-resonant with the atomic transition, $\omega_k \sim \omega_A$. It thus seems reasonable to restrict the frequency integral in Eq.(3.34) to an interval around ω_A , with a width $\Delta\omega$ that is typically smaller than ω_A .

Now, from the properties of the Fourier transformation, we know that the correlation function $C(\tau)$ given by the integral (3.34) has a “width in time” given by $\Delta\tau \Delta\omega \sim 1$. Since this width is precisely the definition of the correlation time, we have

$$\frac{1}{\omega_A} \leq \frac{1}{\Delta\omega} \sim \tau_c. \quad (3.36)$$

Typical value: a few 10 fs.

3.3.2 Evolution of the atomic operators

We shall work in the Heisenberg picture. Note that operators describing different degrees of freedom (field, atom) commute at equal times.

Atomic dipole

“eliminate” the field operators by solving their equation of motion:

$$\frac{da_k}{dt} = \frac{i}{\hbar} [H, a_k] = -i\omega_k a_k - ig_k^* \sigma \quad (3.37)$$

$$a_k(t) = a_k(0) e^{-i\omega_k t} - ig_k^* \int_0^t dt' \sigma(t') e^{-i\omega_k(t-t')} \quad (3.38)$$

the “particular solution” contains the “past” of the atomic dipole operator $\sigma(t')$. Insert this into the equation for σ . Let’s look at this equation first:

$$\begin{aligned} \frac{d\sigma}{dt} &= \frac{i}{\hbar} [H, \sigma] = -i\omega_A \sigma + i \sum_k g_k [\sigma^\dagger, \sigma] a_k \\ &= -i\omega_A \sigma + i \sum_k g_k \sigma_3 a_k \end{aligned} \quad (3.39)$$

We see here that the two-level atom leads to nonlinear equations of motion: the operator product $\sigma_3 a_k$ appears. For the moment, these operators commute (at equal times). But we now want to insert the solution (3.38) for $a_k(t)$, and the two terms that appear here do not commute separately with σ_3 . For this reason, we take now a specific operator order (so-called “normal order”) where the annihilation operators (a_k or σ) act first. This is the order already used in Eq.(3.39), and we thus get

$$\frac{d\sigma}{dt} = -i\omega_A \sigma + i\sigma_3 \mathcal{E}_0(t) + \int_0^t dt' \sum_k |g_k|^2 e^{-i\omega_k(t-t')} \sigma_3(t) \sigma(t') \quad (3.40)$$

where we used the abbreviation

$$\mathcal{E}_0(t) = \sum_k g_k a_k(0) e^{-i\omega_k t} \quad (3.41)$$

for the freely evolving electric field operator (positive frequency component only, and re-scaled by the atomic transition dipole). In the integral over t' , we see that the correlation function $C(t - t')$ appears, and using $\tau = t - t'$ as integration variable, we have

$$\frac{d\sigma}{dt} = -i\omega_A\sigma + i\sigma_3\mathcal{E}_0(t) + \int_0^t d\tau C(\tau)\sigma_3(t)\sigma(t - \tau) \quad (3.42)$$

Now comes the *key observation*: under the time integral occur two very different functions. The correlation function $C(\tau)$ is very narrow in τ . The atomic operator $\sigma(t - \tau)$ contains a “fast free evolution” (generated by the first term in its equation of motion), but once this is factored out, we anticipate that its evolution is “slow”:

$$\sigma(t - \tau) = e^{-i\omega_A(t-\tau)}\tilde{\sigma}(t - \tau) \approx e^{-i\omega_A(t-\tau)}\tilde{\sigma}(t) = e^{+i\omega_A\tau}\sigma(t) \quad (3.43)$$

Here, we introduced temporarily the dipole operator $\tilde{\sigma}(t')$ (in an interaction picture).

The main benefit of this approximation is that only atomic operators at time t appear in the equation of motion. The two-level commutation rules¹ give $\sigma_3(t)\sigma(t) = -\sigma(t)$, and we end up with

$$\frac{d\sigma}{dt} = -i\omega_A\sigma + i\sigma_3\mathcal{E}_0(t) - \sigma(t) \int_0^t d\tau C(\tau) e^{i\omega_A\tau} \quad (3.44)$$

The upper integration limit t is actually the difference between our initial time and the actual time where the equation of motion is computed. We now let this time difference be much larger than the correlation time τ_c . This is consistent with the assumption that the atomic dynamics is slow on the scale of the field’s correlation time. Then the integrand is effectively zero at the upper limit, and we get a constant complex number

$$\gamma + i\delta\omega_A = \int_0^{t \gg \tau_c} d\tau C(\tau) e^{i\omega_A\tau} = \frac{S(\omega_A)}{2} + i\mathcal{P} \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi} \frac{S(\omega)}{\omega - \omega_A} \quad (3.45)$$

where $S(\omega)$ is the Fourier transform of the correlation function $C(\tau)$ and \mathcal{P} means the principal part of the integral.

Spontaneous decay rate and Lamb shift

Explicit calculation: spectrum of vacuum fluctuations

$$S(\omega) = 2\pi \sum_k |g_k|^2 \delta(\omega_k - \omega) \quad (3.46)$$

¹A pedantic remark: operator products evolve as products under the Heisenberg equations of motion. This is because taking the commutator with a product is compatible with the product rule of (time) derivatives.

Take a plane wave expansion and sum over the polarization vectors $\mathbf{u}_{\mathbf{k}\lambda}$ in the coupling constants g_k

$$\sum_{\lambda} |\mathbf{d}_{\text{eg}} \cdot \mathbf{u}_{\mathbf{k}\lambda}|^2 = |\mathbf{d}_{\text{eg}}|^2 - |\mathbf{d}_{\text{eg}} \cdot \hat{\mathbf{k}}|^2 \quad (3.47)$$

where $\hat{\mathbf{k}}$ is the unit vector along \mathbf{k} . This formula arises because the $\mathbf{u}_{\mathbf{k}\lambda}$ are perpendicular to \mathbf{k} . Integration over the angles of \mathbf{k} gives

$$\int d\Omega(\hat{\mathbf{k}}) (|\mathbf{d}_{\text{eg}}|^2 - |\mathbf{d}_{\text{eg}} \cdot \hat{\mathbf{k}}|^2) = 4\pi|\mathbf{d}_{\text{eg}}|^2 - \frac{4\pi}{3}|\mathbf{d}_{\text{eg}}|^2 = \frac{8\pi}{3}|\mathbf{d}_{\text{eg}}|^2 \quad (3.48)$$

The integral over the length of \mathbf{k} is trivial because of the δ -function in Eq.(3.46), its length is fixed to $|\mathbf{k}| = \omega/c$. Putting everything together, we get

$$S(\omega) = \frac{2\pi}{3\hbar^2} |\mathbf{d}_{\text{ge}}|^2 \frac{\hbar\omega}{2\varepsilon_0} \frac{8\pi\omega^2}{(2\pi c)^3} \quad (3.49)$$

where the last factor is the density of field modes per $d\omega$ and volume. We can thus say that the spontaneously decaying atom is a “detector” for vacuum field fluctuations. The decay rate γ

$$\gamma = \frac{1}{2} S(\omega_A) = \frac{|\mathbf{d}_{\text{ge}}|^2 (\omega_A/c)^3}{6\pi\hbar\varepsilon_0} \quad (3.50)$$

is also called the “natural linewidth” of the atomic transition $|g\rangle \leftrightarrow |e\rangle$ because it gives the width in frequency of the spontaneous emission spectrum. In order of magnitude, with $|\mathbf{d}_{\text{ge}}| \sim ea_0$ (electron charge \times atom size)

$$\frac{\gamma}{\omega_A} \sim \alpha_{\text{fs}} (a_0/\lambda_A)^2 \sim \alpha_{\text{fs}}^3 \quad (3.51)$$

with fine structure constant $\alpha_{\text{fs}} = e^2/4\pi\varepsilon_0\hbar c \approx 1/137.04$ and wavelength λ_A of resonant transition. Hence, indeed decay is “slow” on the scale of the optical period.

Frequency shift $\delta\omega_A$ is related to asymmetry of vacuum spectrum around the transition frequency. Interpretation from second order perturbation theory: modes slightly below the atomic resonance, $\omega_k \leq \omega_A$ tend to push the level $|e\rangle$ upwards. Modes above resonance: push downwards. Null effect if spectrum is flat.

Actual calculation requires UV cutoff ω_{uv} and interaction Hamiltonian beyond the resonance (rotating wave) approximation (Hans Bethe \sim 1948, calculation of the Lamb shift). Order of magnitude:

$$\delta\omega_A \approx \gamma \log(\omega_{\text{uv}}/\omega_A) \quad (3.52)$$

and a consistent cutoff is the electron rest mass, $\omega_{\text{uv}} \sim m_e c^2/\hbar$. Hence comparable to the natural linewidth. Note: relativistic theory required, where wavelengths up to electron Compton wavelength $\hbar/m_e c$ are involved. This goes beyond the “long wavelength approximation” because $\hbar/m_e c \sim \alpha_{\text{fs}} a_0 \ll a_0$.

Atomic populations

Actually, we did not yet show that γ is the rate of decay for the excited state. For this, we need the equation of motion for the occupations of the two energy levels. This is described by the atomic operator σ_3 , also called the “inversion” because $\langle \sigma_3 \rangle > 0$ when the excited state is more occupied than the ground state.

Heisenberg equation of motion

$$\begin{aligned} \frac{d\sigma_3}{dt} &= +2i \sum_k \left[g_k^* a_k^\dagger \sigma - g_k \sigma^\dagger a_k \right] \\ &= 2i \left[\mathcal{E}_0^\dagger(t) \sigma - \sigma^\dagger \mathcal{E}_0(t) \right] \\ &\quad - 2 \int_0^t dt' \left[C^*(t-t') \sigma^\dagger(t') \sigma(t) + C(t-t') \sigma^\dagger(t') \sigma(t) \right] \end{aligned} \quad (3.53)$$

where we inserted the formal solution for $a_k(t)$ and brought the operator products in normal order. We apply to the t' -integral the same prescription as before and get

$$\begin{aligned} &\int_0^t dt' \left[C^*(t-t') \sigma^\dagger(t') \sigma(t) + C(t-t') \sigma^\dagger(t') \sigma(t) \right] \\ &\approx (\gamma - i\delta\omega_A) \sigma^\dagger(t) \sigma(t) + (\gamma + i\delta\omega_A) \sigma^\dagger(t) \sigma(t) = \gamma(\sigma_3(t) + \mathbb{1}) \end{aligned} \quad (3.54)$$

where the frequency shift drops out. By construction, the operator $\sigma_3(t) + \mathbb{1}$ gives the occupation of the excited state. From Eqs.(3.53, 3.54), we can thus read off the decay rate 2γ for the excited state population, while the ground state remains stable.

Finally, the equation for the inversion operator becomes

$$\frac{d\sigma_3}{dt} = -2\gamma(\sigma_3(t) + \mathbb{1}) + 2i \left[\mathcal{E}_0^\dagger(t) \sigma - \sigma^\dagger \mathcal{E}_0(t) \right] \quad (3.55)$$

Combined with the equation of motion for the atomic dipole operator,

$$\frac{d\sigma}{dt} = -(\gamma + i\omega_A) \sigma + i\sigma_3 \mathcal{E}_0(t) \quad (3.56)$$

we have thus found the “optical Bloch equations”.

In Eq.(3.56), we have made the replacement $\omega_A + \delta\omega_A \mapsto \omega_A$ for the atomic frequency. This is called “renormalization”: we combine the shift induced by the coupling to the vacuum field with the “naked” transition frequency into the frequency that can be physically observed. Recall that in reality, we can never “switch off” the coupling to the vacuum. Hence the “naked” two-level atom that we started with is actually a theoretical artefact.

3.3.3 Application 1: Bloch equations

Note that we did not specify yet the state of the light field: it is in fact encoded in the operator $\mathcal{E}_0(t)$ that depends on the initial field operators $a_k(0)$. Two examples will be studied now, the first one being an atom driven by a laser field. We have argued that to a good approximation, we can assume that the light field is in a coherent state $|\alpha_L\rangle$. We assume that at $t = 0$, the total system is in the product state $|\psi(0), \alpha_L\rangle$ and take the expectation value of the Bloch equation. This gives the dynamics of the Bloch vector as follows (optical Bloch equations in the proper sense)

$$\frac{ds}{dt} = -(\gamma + i\omega_A)s + (i/2)s_3\Omega e^{-i\omega_L t} \quad (3.57)$$

$$\frac{ds_3}{dt} = -2\gamma(s_3(t) + 1) + i \left[\Omega^*(t)e^{i\omega_L t}s - s^*\Omega e^{-i\omega_L t} \right] \quad (3.58)$$

where $\Omega/2 = \langle \alpha_L | \mathcal{E}_0(0) | \alpha_L \rangle$ is the (complex) Rabi frequency and ω_L the frequency of the laser mode.

These equations have time-dependent coefficients, but this can be removed by making a transformation into a “rotating frame”. We make the replacement

$$s(t) \mapsto s(t)e^{-i\omega_L t} \quad (3.59)$$

where the “new” $s(t)$ satisfies Bloch equations with time-independent coefficients

$$\boxed{\begin{aligned} \frac{ds}{dt} &= -(\gamma - i\delta)s + (i/2)s_3\Omega \\ \frac{ds_3}{dt} &= -2\gamma(s_3(t) + 1) + i[\Omega^*s - s^*\Omega] \end{aligned}} \quad (3.60)$$

where $\delta = \omega_L - \omega_A$ is the “laser detuning”. Note that the symbols are not the same throughout the books: the Rabi frequency Ω or the decay rate γ can differ by a factor of 2 (or -2), the detuning can have the opposite sign.

The Bloch equations are a “workhorse” of atomic physics and quantum optics. They are used to compute light absorption, excitation spectra, population transfer, radiation forces on atoms etc. In the exercises, you compute the stationary state of the Bloch equations (attention with the different sign for δ : this one should be correct)

$$s_{ss} = \frac{-i(\Omega/2)(\gamma + i\delta)}{\gamma^2 + \delta^2 + \Omega^2/2} \quad (3.61)$$

$$s_{3,ss} = \frac{-(\gamma^2 + \delta^2)}{\gamma^2 + \delta^2 + \Omega^2/2} \quad (3.62)$$

Discussion: average atomic dipole operator (induced by laser field), average inversion. Line broadening.

Exercise: total excitation N , does not commute when laser included. Expectation value of \dot{N} in stationary state, interpretation as total scattered intensity.

Exercise: spectrum of spontaneous emission, from formal solution. Need formal solution of atomic dipole operator, Eq.(3.67).

3.3.4 Application 2: the Glauber photodetector

Roy Glauber (Nobel prize 2005) developed in the 1960s the theory of photodetection. His main result is that the signal of a photodetector is proportional to

$$I(t) \propto \langle E^{(-)}(t)E^{(+)}(t) \rangle \quad (3.63)$$

where $E^{(+)}(t)$ is the positive frequency part of the electric field operator at the detector position. This signal is constructed in such a way that if the field is in the vacuum state, a detector gives no signal: perfectly reasonable. But due to the presence of vacuum fluctuations (also nonzero expectation values of a product of field operators!), not easy to implement in the theory.

We can recover the relevant features of Glauber's theory with our two-level atom. We shall actually show that under suitable approximations,

$$I(t) \propto \langle \mathcal{E}_0^\dagger(t)\mathcal{E}(t) \rangle \quad (3.64)$$

where $\mathcal{E}(t)$ is the re-scaled electric field operator we introduced in Eq.(3.41). Note that it contains positive frequency components only, and also only those modes that are near-resonant with the atomic transition frequency ω_A . Indeed, Glauber's model for a photodetector is a two-state system that is prepared in the ground state. Incident light can be absorbed, leading to some population in the excited state. This population is then "rapidly removed" from the system. A physical example: the excited atom is ionized and the free electron moves away (it cannot come back to recombine into the ground state). This is actually the process that happens in a photomultiplier ("avalanche photodiode").

So let us see what our Heisenberg equations (3.55, 3.56) give when the atom is in the ground state and the field is in an arbitrary state. We are interested in the rate of change of the excited state population:

$$I := \frac{dp_e}{dt} = \frac{d}{dt} \left\langle \frac{\sigma_3 + \mathbb{1}}{2} \right\rangle = \frac{1}{2} \frac{d\langle \sigma_3 \rangle}{dt} \quad (3.65)$$

Since the atom is in the ground state, the operator $\sigma_3 + \mathbb{1}$ that appears in Eq.(3.55) averages to zero. We are left with

$$I = i\langle \mathcal{E}_0^\dagger(t)\sigma(t) - \sigma^\dagger(t)\mathcal{E}_0(t) \rangle \quad (3.66)$$

and insert the formal solution for the atomic dipole operator [similar to Eq.(3.38)]:

$$\sigma(t) = \sigma(0) e^{-(\gamma+i\omega_A)t} + i \int_0^t dt' \sigma_3(t') \mathcal{E}_0(t') e^{-(\gamma+i\omega_A)(t-t')} \quad (3.67)$$

This gives

$$\begin{aligned} I &= - \int_0^t dt' \left[\langle \mathcal{E}_0^\dagger(t) \sigma_3(t') \mathcal{E}_0(t') \rangle e^{-(\gamma+i\omega_A)(t-t')} + \text{h.c.} \right] \\ &= \int_0^t d\tau \left[\langle \mathcal{E}_0^\dagger(t) \mathcal{E}_0(t-\tau) \rangle e^{-(\gamma+i\omega_A)\tau} + \text{h.c.} \right] \end{aligned} \quad (3.68)$$

where in the second line, we used the approximation that the inversion $\sigma_3(t')$ is evolving slowly and took its expectation value in the initial state (atom in the ground state, not correlated with the field). If we also assume that the expectation value evolves slowly with time t , $\langle \mathcal{E}_0^\dagger(t) \mathcal{E}_0(t-\tau) \rangle \approx \langle \mathcal{E}_0^\dagger(t+\tau) \mathcal{E}_0(t) \rangle$ [this is strictly true for a stationary field, but stationary fields do not give not very interesting signals on a photodetector], we can combine the ‘+h.c.’ term into the single integral

$$I = \int_{-t}^t dt' \langle \mathcal{E}_0^\dagger(t) \mathcal{E}_0(t+\tau) \rangle e^{-\gamma|\tau|+i\omega_A\tau} \quad (3.69)$$

We observe that the photodetector signal is similar to the Fourier transform of the two-time field correlation function. We can already confirm that the signal is given by a normally ordered expectation value of field operator. Let us consider two limits.

Narrow-band detector

If the decay time $1/\gamma$ is “very long” (compared to the correlation time of the field), the photodetector signal is essentially given by the field spectrum, taken at the transition frequency ω_A . Indeed, the τ -integral in Eq.(3.69) picks from the positive frequency operator $\mathcal{E}_0(t+\tau)$ those components that evolve like $e^{-i\omega_A\tau}$. A narrow band photodetector is thus simply a ‘spectrometer’, and the quantity it measures is the Fourier transform of the temporal correlation (or coherence) function. Experimentally, this situation can be achieved by placing a narrow-band frequency filter in front of a photodetector.

Fast detector

In some photodetectors, the excited state has a very short lifetime $1/\gamma$. This happens for example when the excited state actually ionizes and gives away its electron. The

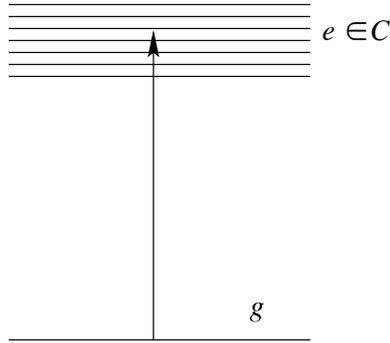


Figure 3.4: Two-level model for a fast photodetector. The signal corresponds to a sum of transition rates into all components of the upper state (quasi)continuum.

excitation rate I is then actually an integral over all components of the excited state ‘continuum’, as sketched in Fig.3.4. Under the integral, the exponential $e^{-\gamma|\tau|}$ becomes very short-ranged in τ , and we can make the replacement

$$e^{-\gamma|\tau|} \mapsto \frac{2}{\gamma} \delta(\tau) \quad (3.70)$$

This leads to Glauber’s formula

$$I(t) = \eta \langle \mathcal{E}_0^\dagger(t) \mathcal{E}_0(t) \rangle \quad (3.71)$$

where the expectation value of the instantaneous intensity operator $\mathcal{E}_0^\dagger(t) \mathcal{E}_0(t)$ appears, in normal order, of course. The prefactor η actually is a number characteristic for the detector and is called “quantum efficiency”. It must be determined experimentally. If the intensity is scaled to “photons per second”, then η gives the detection probability per photon, and I the “rate of detected photons per second”.

Intensity (photon) correlations

Generalization to multiple detection events: $I(t)$ interpreted as probability per unit time to detect one photon at time t . Probability to detect one photon at time t_1 and another one at t_2 :

$$I(t_1, t_2) = \eta^2 \langle \mathcal{E}_0^\dagger(t_1) \mathcal{E}_0^\dagger(t_2) \mathcal{E}_0(t_2) \mathcal{E}_0(t_1) \rangle, \quad t_2 > t_1 \quad (3.72)$$

time- and normal-ordered operator product. Interpretation.

Temporal coherence

Definition in terms of factorization of $\langle \mathcal{E}_0^\dagger(t) \mathcal{E}_0(t') \rangle$ (first-order temporal coherence). And for second-order.

Examples: monochromatic field always first-order coherent. Black-body radiation not. Two-photon state is second-order coherent, but not in first-order.

Bibliography

- R. J. Adler, B. Casey & O. C. Jacob (1995). Vacuum catastrophe: An elementary exposition of the cosmological constant problem, *Am. J. Phys.* **63** (7), 620–26. Comment by S. R. Ovshinsky and H. Fritzsche, *Am. J. Phys.* 65 (1997) 927 and reply p. 928.
- P. R. Berman & V. S. Malinovsky (2011). *Principles of Laser Spectroscopy and Quantum Optics*. Princeton University Press, Princeton Oxford.
- C. Cohen-Tannoudji, J. Dupont-Roc & G. Grynberg (1987). *Photons et atomes — Introduction à l'électrodynamique quantique*. InterEditions, Paris. English translation: *Photons and Atoms — Introduction to Quantum Electrodynamics* (Wiley, New York 1989).
- D. P. Craig & T. Thirunamachandran (1984). *Molecular Quantum Electrodynamics*. Academic Press, London.
- J. Dalibard, Y. Castin & K. Mølmer (1992). Wave-Function Approach to Dissipative Processes in Quantum Optics, *Phys. Rev. Lett.* **68**, 580–583.
- P. A. M. Dirac (1927). The Quantum Theory of the Emission and Absorption of Radiation, *Proc. Roy. Soc. (London) A* **114**, 243–65.
- H. Haken & H. C. Wolf (2000). *The Physics of Atoms and Quanta*. Advanced Texts in Physics. Springer, Heidelberg Berlin, 6th edition.
- C. Henkel (2007). Laser theory in manifest Lindblad form, *J. Phys. B: Atom. Mol. Opt. Phys.* **40**, 2359–71.
- C. Itzykson & J. B. Zuber (2006). *Quantum field theory*. Dover, New York.
- R. Loudon (2000). *The Quantum Theory of Light*. Oxford University Press, Oxford, 3rd edition.
- L. Mandel & E. Wolf (1995). *Optical coherence and quantum optics*. Cambridge University Press, Cambridge.

- P. Meystre & M. Sargent III (1999). *Elements of Quantum Optics*. Springer, Berlin, 3rd edition.
- A. B. Migdal (1977). *Qualitative methods in quantum theory*. Perseus Publishing, Cambridge, Mass.
- M. A. Nielsen & I. L. Chuang (2000). *Quantum Computation and Quantum Information*. Cambridge University Press, Cambridge.
- W. H. Zurek (1991). Decoherence and the transition from quantum to classical, *Physics Today* **44** (10), 36–44. Update: [quant-ph/0306072](#).