Einführung in die Quantenoptik I

Wintersemester 2013/14
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Problem 2.1 – About the size of a photon (6 points)

Quantum opticians are still discussing how strongly a single photon can be spatially localized. The central concept remains the same, however: a 'photon' with (mean) frequency ν has an energy $h\nu$ that is the space integral of the electromagnetic energy density $u = \frac{1}{2}\varepsilon_0 \mathbf{E}^2 + \frac{1}{2}\mu_0 \mathbf{H}^2$ over the region where the photon is 'localized'.

Take a photon in the visible range and estimate the strength of its electric and magnetic fields (**H** in $\mathrm{A/m}$ and **B** in T) for the following three scenarios: (1) photon completely delocalized over the volume of a Fabry-Pérot cavity (ask in the photonics lab about the typical size); (2) photon localized within the size of a wavelength (volume λ^3); (3) photon localized within the size of the Compton wavelength of an electron $\lambda_e = \hbar/mc$ (Ole Keller, Aalborg, "Quantum Theory of Near Field Electrodynamics" 2011).

Problem 2.2 – Rate equations and thermalization (8 points)

In the lecture, we have seen the rate equations that Einstein introduced for the occupations $p_{\rm g}$, $p_{\rm e}$ of a two-level system:

$$\frac{\mathrm{d}p_{\rm e}}{\mathrm{d}t} = R p_{\rm g} - \gamma p_{\rm e} \tag{2.1}$$

(i) Write down the rate equation for $p_{\rm g}$, knowing that $p_{\rm g} + p_{\rm e} = 1$ is a conserved quantity. (Why?)

(ii) Describe in words the meaning of the rates R and γ.

(iii) Solve the rate equations in the steady state where $\mathrm{d}p_a/\mathrm{d}t = 0$. Calculate the ratio of the rates R and γ by imposing that in this steady state, the ratio of the populations is given by the Boltzmann factor ($k_{\rm B} = 1$)

$$\frac{p_{\rm e}}{p_{\rm g}} = \mathrm{e}^{-(E_{\rm e} - E_{\rm g})/T} \tag{2.2}$$

According to Bohr, the energy difference $E_{\rm e} - E_{\rm g} = \hbar\omega_A$ contains the frequency ω_A of photons that can be absorbed and emitted by the atom.

(iv) Einstein told us that the rate R is proportional to the average photon number, $R = B\bar{N}$ (absorption), while $\gamma = A + B\bar{N}$ (spontaneous and stimulated emission, Einstein's famous A and B coefficients). Calculate $\bar{N}$ from Eq.(2.2) and compare to Planck's formula for the average photon number in thermal equilibrium. What would you expect for the ratio B/A?

(v) [5 bonus points] Numerically solve the rate equations for system initially in an arbitrary state. Make a plot including the steady-state formula and the analytical solution.

Problem 2.3 – Thermalization and information (6 points)

Consider a quantum system with discrete states described by the density matrix ρ (a hermitean, positive definite matrix with trace $\mathrm{tr}\,\rho = 1$). The *von Neumann entropy* of ρ is defined by analogy to the Shannon information as

$$S_{\mathrm{vN}}(\rho) = -\sum_i p_i \log p_i =: -\mathrm{tr}\,(\rho \log \rho) \tag{2.3}$$

where p_i are the eigenvalues of ρ. (i) Show that $S_{\mathrm{vN}}(\rho) \le \log 2$ for a two-state system; the maximum is reached if both states occur with equal probability.

(ii) Show that $U\rho U^\dagger$ has the same entropy as ρ where U is a unitary matrix. Conclude that for a closed system, entropy is conserved.

(iii) The *thermodynamical entropy* is given in terms of the canonical partition function Z (*kanonische Zustandssumme*) by

$$S_{\mathrm{td}} = \frac{\partial}{\partial T}(T \log Z) \tag{2.4}$$

Show that this entropy coincides with the von Neumann entropy for a system in thermal equilibrium. Calculate this quantity for a two-level system and sketch it as a function of temperature.

Note. The partition function in the canonical ensemble is defined by $Z = \mathrm{tr}\,\exp(-H/T)$ where T is the temperature and H the system Hamiltonian (a hermitean, positive semi-definite matrix). In thermal equilibrium, the system density matrix is $\rho = Z^{-1}\,\exp(-H/T)$.