Einführung in die Quantenoptik I

Wintersemester 2013/14
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Problem 5.1 – Stationary solution of the Bloch equations (20 points)

[+10 bonus points] In the lecture, we have seen the Bloch equations in the resonance approximation. For a few selected elements of the density matrix:

$$\frac{d\rho_{\rm ee}}{dt} = -\gamma\rho_{\rm ee} + \frac{\rm i}{2}\left(\Omega^*\rho_{\rm eg} - \Omega\rho_{\rm ge}\right) \tag{5.1}$$

$$\frac{d\rho_{\rm eg}}{dt} = -(\Gamma - {\rm i}\Delta)\rho_{\rm eg} + {\rm i}\frac{\Omega}{2}\left(\rho_{\rm ee} - \rho_{\rm gg}\right) \tag{5.2}$$

This is written in the 'rotating frame' (for the slow variables), but we dropped the tilde $\tilde{\rho}$.

(1) Explain how you get the equations of motion for the other matrix elements.

(2) Describe what are the 'Hamiltonian parts' and the 'dissipative parts' in the Bloch equations. Check that the Hamiltonian parts arise from the (effective) Hamiltonian

$$H_{\rm AL} = -\frac{\hbar\Delta}{2}\sigma_3 + \frac{\hbar}{2}\left(\Omega^*\sigma + \Omega\sigma^\dagger\right) \tag{5.3}$$

where σ and $\sigma^\dagger$ are the ladder operators (down and up) for the two-level system and σ_3 the Pauli matrix.

(3) Since the Bloch equations include damping, one finds a unique stationary state $\rho(\infty)$, i.e., a density matrix whose time derivative is zero. Calculate this state and show that the imaginary part of $\rho_{\rm eg}(\infty)$ is given by a Lorentz curve as a function of laser frequency $\omega_L = \omega_A + \Delta$. Show that the line width of this Lorentz curve becomes larger when the Rabi frequency Ω increases: this is called *power broadening* or *saturation*.

(4) Plot for the stationary state the absolute value of $\rho_{\rm eg}(\infty)$ and $\rho_{\rm ee}(\infty)$ for zero detuning ($\omega_L = \omega_A$) as a function of the Rabi frequency Ω. In the literature, these two quantities are called 'coherent' and 'incoherent scattering'. Find out why these names are given.

Remember: in the stationary state, the dipole moment of the two-level atom oscillates at frequency ω_L. Hint: the coherent scattering goes through a maximum.

(5*) [Bonus points] In the resonance approximation, there is a simple relation between the elements $\tilde{\rho}_{ab}(t)$ and $\rho_{ab}(t)$ of the density matrix ($a, b = \mathrm{e}, \mathrm{g}$). Form the density matrix $\tilde{\rho}(t)$ and show that it is related to the original density matrix $\rho(t)$ by a conjugation:

$$\tilde{\rho}(t) = \exp(-\mathrm{i}\omega_L t \sigma_3/2) \rho(t) \exp(\mathrm{i}\omega_L t \sigma_3/2) \tag{5.4}$$

Check the following rules for the expectation value of the dipole operator

$$\langle \sigma_1 \rangle = \mathrm{tr}\left[\left(\sigma\, \mathrm{e}^{-\mathrm{i}\omega_L t} + \sigma^\dagger\, \mathrm{e}^{\mathrm{i}\omega_L t} \right) \tilde{\rho}(t) \right] = \tilde{\rho}_{\mathrm{eg}}\, \mathrm{e}^{-\mathrm{i}\omega_L t} + \tilde{\rho}_{\mathrm{ge}}\, \mathrm{e}^{\mathrm{i}\omega_L t} \tag{5.5}$$

Remember from the mathematical exercise (Problem 4.1) that Eq.(5.4) has a geometric interpretation for the Bloch vectors s and $\tilde{\mathrm{s}}$ corresponding to ρ and $\tilde{\rho}$: they are related by a rotation around the 3-axis.

(6*) [Bonus points] Remember how you found for an oscillator with an external force (frequency ω different from resonance frequency ω_0) the 'long-time limit' for position x and momentum p, after the transients have decayed. Where in the discussion do you need the damping?

(7*) [Bonus points] Use the procedure of the lecture for separating fast and slow time scales and find an equation of motion for the so-called 'anti-resonant terms' in the density matrix:

$$\begin{aligned}
\rho_{\mathrm{ee}}(t) &= \overbrace{\tilde{\rho}_{\mathrm{ee}}(t)}^{\text{slow}} + \overbrace{A(t)\cos(2\omega_L t - \phi(t))}^{\text{anti-resonant}} && (5.6)\\
\rho_{\mathrm{eg}}(t) &= \underbrace{\tilde{\rho}_{\mathrm{eg}}(t)}_{\text{slow}}\, \mathrm{e}^{-\mathrm{i}\omega_L t} + \underbrace{B(t)\, \mathrm{e}^{\mathrm{i}\omega_L t}}_{\text{anti-resonant}} && (5.7)
\end{aligned}$$

You are allowed to make the approximation that $A(t)$, $\phi(t)$, $B(t)$ are slowly varying over one laser period $2\pi/\omega_L$. A complex amplitude $A(t)\,\mathrm{e}^{\mathrm{i}\phi(t)}$ is also a useful quantity to look at.