Einführung in die Quantenoptik I

Wintersemester 2013/14
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Problem 6.1 – Macroscopic Bloch equations (12 points)

In the lecture, we have seen so far the Bloch equations for the state of a *single* two-level system (Pauli matrices σ, $\sigma^\dagger$, σ_3). In many applications, one has to describe a medium with many dipoles (density N = number/volume). For simplicity, we assume that the density N is spatially constant and does not depend on time (the dipoles do not move).

(1) Recall the concept of a polarization field $\mathbf{P}(\mathbf{r}, t)$ and justify the following formula

$$\mathbf{P}(\mathbf{r}, t) = N\mathbf{d}\langle\sigma_1(t; \mathbf{E}(\mathbf{r}, t))\rangle \tag{6.1}$$

where $\mathbf{d} = \mathbf{d}_{\rm eg}$ is the matrix element of the dipole operator. The purpose of the notation $(\ldots; \mathbf{E}(\mathbf{r}))$ is to remember that the expectation value is calculated from the electric field $\mathbf{E}(\mathbf{r}, t)$ at the position $\mathbf{r}$.

No confusion between the polarization field and the polarization vector of the electromagnetic field.

(2) Recall from electrodynamics the wave equation

$$-\nabla \times \nabla \times \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial \mathbf{j}}{\partial t} \tag{6.2}$$

and look up the link between current density and polarization field.

(3) Consider a stationary regime where all fields in Eq.(6.2) oscillate at the frequency ω_L and with a wave vector $\mathbf{k}$:

$$\mathbf{E}(\mathbf{r}, t) = \vec{\mathcal{E}} \exp{\rm i}(\mathbf{k}\cdot\mathbf{r} - \omega_L t) + \text{c.c.} \tag{6.3}$$

Introduce the polarizability

$$\mathbf{d}\langle\sigma_1(t; \mathbf{E}(\mathbf{r}, t))\rangle = \alpha(\omega_L)\vec{\mathcal{E}} \exp{\rm i}(\mathbf{k}\cdot\mathbf{r} - \omega_L t) + \text{c.c.} \tag{6.4}$$

and give the expression for $\alpha(\omega_L)$ from the Bloch equations.

(4) Justify the following result for the dispersion relation of light in the medium

$$k^2 = \frac{\omega_L^2}{c^2}\varepsilon(\omega_L) \tag{6.5}$$

and calculate the dielectric function of the medium

$$\varepsilon(\omega_L) = 1 + \frac{N\alpha(\omega_L)}{\varepsilon_0} \tag{6.6}$$

Find an order of magnitude for this dimensionless number at the resonance frequency of the medium.

Problem 6.2 – Kubo formula for polarizability (8 points)

A key result of time-dependent perturbation theory is the Kubo formula which gives the response function $\chi(t)$ of a system to a weak perturbation. If we specialize this a polarizable atom, the response of the electric dipole $\hat{\mathbf{d}}(t)$ to a weak electric field $\mathbf{E}(t)$ can be written in the form

$$\langle \hat{\mathbf{d}}(t) \rangle = \int_{-\infty}^{+\infty} \mathrm{d}\tau\, \chi_{dE}(\tau)\mathbf{E}(t-\tau) \tag{6.7}$$

where $\chi_{dE}(\tau)$ is given by (no need to prove this)

$$\chi_{dE}(t-t') = \frac{\mathrm{i}}{\hbar}\Theta(t-t')\langle[\hat{\mathbf{d}}(t), \hat{\mathbf{d}}(t')]\rangle_0 \tag{6.8}$$

where the expectation value $\langle \ldots \rangle_0$ is taken in the non-perturbed state. The operator $\hat{\mathbf{d}}(t)$ evolves in the Heisenberg picture without an external field (= no perturbation).

(1) Justify the step function $\Theta(t-t')$ in Eqs.(6.7, 6.8).

(2) Use Eq.(6.8) to calculate the polarizability $\alpha_{\mathrm{g}}(\tau)$ in the ground state and its Fourier transform $\alpha_{\mathrm{g}}(\omega)$.

(3) [5 Bonus points.] Calculate the polarizability in a thermal state, i.e. a density operator where the levels $|\mathrm{g}\rangle$ and $|\mathrm{e}\rangle$ are occupied with Boltzmann weights.