Einführung in die Quantenoptik I

Wintersemester 2013/14
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Problem 7.1 – Field commutators (10 points)

In the lecture, we have seen the commutator between the fields operators $\mathbf{E}(x)$ and $\mathbf{B}(x)$:

$$[E_j(\mathbf{x},t),\, B_k(\mathbf{x}',t)] = (\text{const.})\mathrm{i}\hbar\epsilon_{jkl}\frac{\partial}{\partial x_l}\delta(\mathbf{x}-\mathbf{x}') \tag{7.1}$$

where the constant depends on the system of units ($-1/\varepsilon_0$ in SI units). All other commutators vanish. All fields in this problem have to be understood as operators.

(1) Fix the constant in Eq.(7.1) from the following 'correspondence principle': in the Heisenberg picture, the standard equation of motion for the magnetic field

$$\frac{\partial}{\partial t}\mathbf{B} = \frac{\mathrm{i}}{\hbar}\,[H,\,\mathbf{B}]$$

should reduce to the Faraday equation

$$\frac{\partial}{\partial t}\mathbf{B} = -\nabla\times\mathbf{E} \tag{7.2}$$

and the Hamiltonian H is simply the space integral over the field energy

$$H = \int \mathrm{d}^3x \left(\frac{\varepsilon_0}{2}\mathbf{E}^2(x) + \frac{1}{2\mu_0}\mathbf{B}^2(x)\right) \tag{7.3}$$

Hint. Recall that the commutator acts on products similar to the product rule of differential calculus:

$$[H, AB] = [H, A]\,B + A\,[H, B]$$

The distribution 'derivative of δ-function' in Eq.(7.1) is *defined* by the rules of partial integration. You can safely assume that there is no contribution from boundary terms.

(2) Consider the amplitude operators for smooth mode functions $\mathbf{f}$ and g:

$$\mathcal{E} = \int \mathrm{d}^3x\,\mathbf{f}(\mathbf{x})\cdot\mathbf{E}(\mathbf{x}), \qquad \mathcal{B} = \int \mathrm{d}^3x\,\mathbf{g}(\mathbf{x})\cdot\mathbf{B}(\mathbf{x}) \tag{7.4}$$

and assume that the modes are related by $(\omega/c)\mathbf{f} = \nabla\times\mathbf{g}$. Work out the commutator $[\mathcal{E},\mathcal{B}]$ and conclude that orthogonal components of $\mathbf{E}$ and $\mathbf{B}$ at neighboring points cannot be measured simultaneously. Derive, as in the quantum mechanics I course, the uncertainty relation between the variances $(\Delta\mathcal{E})^2$, $(\Delta\mathcal{B})^2$. Use this to show that the electric and magnetic energies in these modes have a quantum uncertainty of the order of $\hbar\omega$.

Problem 7.2 – Photon operators (5 points)

Consider a two-level atom (ladder operators σ, $\sigma^\dagger$) and a single mode of the radiation field with creation and annihilation operators $a^\dagger$, a. Their interaction can be written in the form

$$V = \hbar\Omega_1(a^\dagger\sigma + \sigma^\dagger a) \tag{7.5}$$

By working out the action of V on a typical state $|b, n\rangle$ ($b = e, g$, $n = 0, 1, 2\ldots$), describe in words the physical processes that it generates and comment on the name 'one-photon Rabi frequency' for Ω_1. Show that V commutes with the 'excitation operator'

$$N = a^\dagger a + \sigma^\dagger\sigma = a^\dagger a + \tfrac{1}{2}(\sigma_3 + \mathbb{1}) \tag{7.6}$$

and give a physical interpretation of the eigenstates of N.

Problem 7.3 – Vacuum energy (5 points)

In the energy of the radiation field given in the lecture, we have "neglected" the contribution of the zero-point energy of each field mode, given by $\frac{1}{2}\hbar\omega_k$ per mode. This 'small term' leads, in the vacuum state of the electromagnetic field, to an energy density u_{vac} that depends on the cutoff momentum k_c as follows

$$u_{\text{vac}} = C\hbar c k_c^4 \tag{7.7}$$

where C is a numerical constant "of order unity" (we calculate it in the lecture). (i) Fix the cutoff at the energy scale $E_{\text{GUT}} = \hbar c k_c$ for the "unification" of the fundamental interactions (except gravity, look up on the Internet the keyword "grand unified theory") and make an estimate for the corresponding vacuum energy. Express this number in proton masses (times c^2) per cubic meter. (ii) Look up the length scale $\ell_{\text{Planck}} = 1/k_c$ for the unification of gravity, relativity, and quantum theory and compare the corresponding vacuum energy to case (ii). (iii) Somebody told you that the density of "dark energy" in the vacuum (responsible for the accelerated expansion of the Universe, Nobel prize in physics 2011) is of the order of one proton rest mass (times c^2) per cubic meter. Find out more details on this and compare to the energy density of the cosmic microwave background.

The vacuum energy density predicted by quantum electrodynamics (one of the simplest quantum field theories) is *very, very* different from that attributed to dark energy, and there is no good explanation yet for this.