Quantum Information Theory (WiSe 2014/21015)

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Problem Set No 2 ($40 + 2\pi$ scores)[1]

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▷ Aufgabe 1 (Witch-path detection) (20 scores)

We consider a Young double slit experiment for spin-$\frac{1}{2}$ particle. We assume some magnetic field behind one of the slits – the slit on the right side, say. As we shall see the magnetic field – although it has no impact on the motional state of the atoms – may have a severe impact on the interference pattern of the double slit.

The particles impinge in some definite spin state $|\uparrow_z\rangle$. Behind the double slit the state vector reads

$$|\Psi\rangle = \frac{|l\rangle \otimes |\uparrow_z\rangle + |r\rangle \otimes |\uparrow_a\rangle}{\sqrt{2}}\,, \tag{1}$$

where $|l\rangle$ ($|r\rangle$) is the translational state of particles which passed through the left (right) slit (with the other slit closed), and $\uparrow_a$ is the rotated spin state of particles which passed through the right slit. In the position representation the translational states are $l(x) := \langle x|l\rangle \propto e^{ikx}$, $r(x) := \langle x|r\rangle \propto e^{-ikx}$.

The state vector (1) is that of an entangled state, yet it is not written in the form of a Schmidt decomposition. In contrast to the motional state "passage though the left slit" $|l\rangle$ and "passage through the right slit" $|r\rangle$, which are true alternatives, $\langle l|r\rangle = 0$, the spin state are not necessarily orthogonal.

(a) Compute the Schmidt decomposition of Eq. (1). Confirm

$$|\Psi\rangle = \sqrt{p}|\phi_+\rangle \otimes |\uparrow_c\rangle + \sqrt{1-p}|\phi_-\rangle \otimes |\downarrow_c\rangle \tag{2}$$

where $\vec{c} = (\vec{z} + \vec{a})/\|\vec{z} + \vec{a}\|$ is a spatial unit vector, $|\phi_\pm\rangle = (|r\rangle \pm |l\rangle)/\sqrt{2}$. What is p expressed in terms of $\vec{a}, \vec{z}$?

(b) The particles impinge on a detection screen (a CCD camera, say) which is not sensitive to the spin state. Compute the probability density I at point x on the detection screen. Confirm

$$I(x) \propto |l(x)|^2 + |r(x)|^2 + \beta l(x)^* r(x) + \beta^* r(x)^* l(x)\,. \tag{3}$$

where

$$\beta = \langle \uparrow_z | \uparrow_a\rangle\,. \tag{4}$$

(c) The density I diplays an interference pattern, the modulation depth of which, called *fringe contrast*, depends sensitively on the spin-state overlap β. Using the definition of the fringe contrast

$$\gamma := \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} \tag{5}$$

please confirm

$$\gamma = |\beta|\,. \tag{6}$$

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[1] Problems with transcendental scores are facultative nuts. Nuts have high nutrition value ...

For maximally distinguishable spin states $\beta = 0$, the contrast is zero and the interference pattern turns into the distribution of classical point particles. For indistinguishable spin states, $\beta = 1$, contrast is maximal, i.e. the density distribution is "maximally quantum". Note that the reduction in contrast takes place even though, in our model, the particle's motion is not influenced by the magnetic field.

Yet spin measurement may reveal information about the slit each invidual particle took before reaching the screen, called *which-path-information*. The more which-path information we can extract, the lower the fringe contrast, and concomitantly, the more "classical" the density distribution.

(d) Use the rules of elementary quantum mechanics to confirm

$$\mathrm{Prob}(l, \uparrow_z) = \tfrac{1}{2}\,, \qquad \mathrm{Prob}(r, \uparrow_z) = \tfrac{1}{2}q\,, \tag{7}$$

$$\mathrm{Prob}(l, \downarrow_z) = 0\,, \qquad \mathrm{Prob}(r, \downarrow_z) = \tfrac{1}{2}\left(1 - q\right)\,. \tag{8}$$

where $q = |\beta|^2$.

(e) Infer $\mathrm{Prob}(\uparrow_z) = \frac{1}{2}(1 + q)$, $\mathrm{Prob}(\downarrow_z) = \frac{1}{2}(1 - q)$, and the conditional probabilities

$$\mathrm{Prob}(l| \uparrow_z) = \frac{1}{1+q}\,, \qquad \mathrm{Prob}(r| \uparrow_z) = \frac{q}{1+q}\,, \tag{9}$$

$$\mathrm{Prop}(l| \downarrow_z) = 0\,, \qquad \mathrm{Prob}(r| \downarrow_z) = 1\,. \tag{10}$$

(f) The conditional probabilities, in turn, can be quantified in terms of conditional entropies. Cofirm that the residual uncertainty, which remains about the path given the particle is detected with spin up, is given by

$$H(\mathrm{path}| \uparrow_z) = \frac{1}{1+q}\log_2(1+q) + \frac{q}{1+q}\log_2\left(\frac{1+q}{q}\right)\,, \tag{11}$$

while $H(\mathrm{path}| \downarrow_z) = 0$, because $\downarrow_z$ can only be found for particles which took the right slit.

(g) The initial level of ignorance about the path is $H_{\mathrm{initial}} = 1bit$. The average level of ignorance which remains after a spin measurement is $H_{\mathrm{final}} = \mathrm{Prob}(\uparrow_z)H_\uparrow + \mathrm{Prob}(\downarrow_z)H_\downarrow$. The average information gain $I_{\mathrm{av}} = H_{\mathrm{initial}} - H_{\mathrm{final}}$. Compute I_{av} and summarize: Information gain is maximal if the spin states $\uparrow_z$, $\uparrow_a$ are orthogonal, $q = 0$. If the spin states are parallel, $q = 0$, information gain is zero – in this case spin measurement does not reveal any which-path-information.

Evidently, it is the mere presence of which-path information, and not the uncontrolled scattering of a photon, say, which affects the spatial density distribution. The more we can learn about the path, the more classical appears the distribution. The less we can learn about the path, the more quantum appears the distribution.

In our model, the which-path information and fringe contrast is intimately linked to the entanglement between the motional and spin degrees of freedom. The more entanglement, the better the which-path measurement. The better the which-path measuremnt, the less quantum the pattern. The less quantum the pattern, the more classical the distribution. Entanglement may well destroy that what is most important – the coherence. In the present case it destroys the coherence between the wave functions $l(x)$ and $r(x)$.

▷ Aufgabe 2

For a pair of qubits, the Bell basis

$$|\Psi^\pm\rangle := \frac{1}{\sqrt{2}}\left(|\uparrow_z\downarrow_z\rangle \pm |\downarrow_z\uparrow_z\rangle\right)\,, \tag{12}$$

$$|\Phi^\pm\rangle := \frac{1}{\sqrt{2}}\left(|\uparrow_z\uparrow_z\rangle \pm |\downarrow_z\downarrow_z\rangle\right)\,. \tag{13}$$

plays an important role. Here, the first entry refers to Alices' qubit, the second entry refers to Bob's qubit.

(a) Most prominent the singlet $|\Psi^-\rangle$, for which we would like you to prove the invariance under simultaneous rotation of the quantization axis,

$$|\uparrow_a\downarrow_a\rangle - |\downarrow_a\uparrow_a\rangle = |\uparrow_b\downarrow_b\rangle - |\downarrow_b\uparrow_b\rangle\,. \tag{14}$$

(b) Adopting the standard matrix representation (**??**), confirm that $|\Psi^-\rangle$ reads

$$|\Psi^-\rangle \mapsto \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ +1 \\ -1 \\ 0 \end{bmatrix}\,. \tag{15}$$

What about the other basis vectors?

(c) Introduce the state $\hat{\varrho} := |\Psi^-\rangle\langle\Psi^-|$ and confirm the matrix representation,

$$\hat{\varrho} \mapsto \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & +1 & -1 & 0 \\ 0 & -1 & +1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{16}$$

What are the eigenvalues and associate eigenvectors of this matrix (have a look at (15)?

(d) Perform the partial transpose on Bobs qubit; confirm that in the matrix representation, the image reads

$$\hat{\varrho}^{T_B} \mapsto \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

What are the eigenvalues of $\hat{\varrho}^{T_B}$? So – is that still a state?

▷ Aufgabe 3 (Decrypt this!) (π scores)

As an inofficial employee of your countrie's special service you are of course well aware of the relative frequencies of the letters in English language documents (see Fig 1 for a reminder). Hence for you it is a piece of cake to decrypt the following cryptogram

```
,(flphjmkcfj,kslg-zdsfclzplvzcchj8vk,8zjl8
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Letter	Percentage	Letter	Percentage	Letter	Percentage	Letter	Percentage
a	5.75	h	3.13	o	6.89	v	0.69
b	1.28	i	5.99	p	1.92	w	1.19
c	2.63	j	0.06	q	0.08	x	0.73
d	2.85	k	0.84	r	5.08	y	1.64
e	9.13	l	3.35	s	5.67	z	0.07
f	1.73	m	2.35	t	7.06	-	19.28
g	1.33	n	5.96	u	3.34		

Abbildung 1: Distribution (percentage) over the 27 outcomes for a randomly selected letter in an English language document *The Frequently Asked Questions Manual for Linux*. Quoted after: David J.C. MacKay, Cambridge, UK.

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il,(k,lzpl-fg-zmhv8jalk,lzjflgz8j,lf8,(f-l
fnkv,selz-lkgg-zn8ck,fselklcfiikaflifsfv,f
mlk,lkjz,(f-lgz8j,qluvskhmfli(kjjzj.lwrt4o
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which is known to be the result of a simple substitution cipher executed on an English text. What message did the sender of this cryptogram try to conceal?

▷ Aufgabe 4 (Qubit Communication) (6 scores)

Alice prepares a qubit either in state $|1\rangle = |\uparrow_a\rangle$, or in state $|2\rangle = |\downarrow_a\rangle$. The probabilities that she does the one or the other are given by $p_1 = p_2 = 0.5$.

Although Bob knows Alice's choice $\vec{a}$, he measures with orientation $\vec{b}$ (just to see how much he could learn); the two outcomes are labeled $+$ and $-$.

(a) What is the probability $P_{\sigma|i}$ that Bob finds σ, $\sigma = \pm$, given that Alice transmitted $|i\rangle$, $i = 1, 2$?

(b) What is the probability that Bob finds $\sigma = \pm$ irrespective of what Alice transmitted?

(c) What is the probability that Alice in fact transmitted $|i\rangle$ given that Bob found σ?

(d) What orientation should Bob chose for perfect communication? How many bits per qubit are revealed with this orientation?

Background This example proves that in an ideal world, where there is no disturbing influence from the enviroment, 100-percent reliable communication is possible using qubits. The only point sender and receiver must be aware of is to chose equal alignment of their Stern-Gerlach devices. The example also indicates that in this particular context, the maximal amount of information which a single qubit can carry is one bit. Later in the lecture we shall see that in a different context, which involves entanglement, one can cram two bits in a single qubit.

▷ Aufgabe 5 (Daimler-Benz) (10 scores)

Alice prepares a qubit in the up-state $|\uparrow_i\rangle$ with respect to one out of three possible quantization axis $\vec{a}_i$, $i = 1, 2, 3$, where the $\vec{a}_i$ form a co-planar "Mercedes-Stern",

$$\sum_{i=1}^{3} \vec{a}_i = 0\,. \tag{18}$$

Bob knows the possible directions $\vec{a}_i$, but he does not know which particular direction Alice has chosen. What is his initial level of ignorance? How much could he expect to learn about Alice's choice, and what is his optimal strategy?

▷ Aufgabe 6 (So much for eternal truths . . .) (4 scores)

In you first year studies of the computer sciences, you may well be confronted with the fundamental theorem of logic circuit computation

Theorem There is no unary gate which upon two-fold concatenation yields the logical NOT.

Being a smart student, you can easily prove the theorem.

With no challenges left, you decide to quit computer sciences, and attend quantum mechanics classes instead. There you are also confronted with the fundamental theorem of logic circuit computation, which to your surprise reads

Theorem There are infinitely many unary gates, each of which yields upon two-fold concatenation the logical NOT.

Can you prove this theorem? And if so – is'nt physics wonderful?

▷ Aufgabe 7 (PQ-Penny Flip) (π scores)

Take a friend, go to the bar, get a drink and play a game:

Place a coin head up in a box. Seal the box so that nobody can look inside. You will now take three turns, first you, then your friend, then you again. At each turn you (or your friend) can manipulate the coin in any desired manner, for example turn it around, or not turn it around. Of course neither you nor your friend can see the actual state of the coin (heads or tails up). Also, you can't see what action your friend takes (turn or not turn), nor can your friend see what action you take. Once you are done, you may open the box. You win if the coin is still head up in the end. Otherwise your friend wins.

(a) Convince your friend that there is no winning strategy for neither you nor your friend.

(b) Recall quantum mechanics (but don't tell your friend) and win the game – always!

Reference: D. Meyer, Phys. Rev. Lett. **82**, 1052.

▷ Aufgabe 8 (Monty Hall)

Participating in a game show you have luckily reached the final round where you are given the opportunity to collect your prize. The prize is hidden behind one of three doors, all of which are closed. You are asked to point at that door behind which you expect the prize is hidden. At that point, the door is not yet opened, but Monty Hall, the show master (who knows where the prize is hidden), opens another door instead, behind which there is no prize. He then asks you whether you insist on your initial choice or you rather prefer to switch to the remaining door. Once you have announced your decision, the corresponding door is opened, and you may take home whatever is behind that door (i.e. the prize, if you are lucky, or nothing, if you are unlucky). How would you decide?

Background: The problem, which also runs under the name "goat problem" (the prize is a goat), became famous when in the eighties, Marilyn von Savant presented the correct solution in the Scientific American. Her solution was fiercly attacked by even the most prestigeous experts in statistical analysis, who – in the end – were all wrong and Marilyn was all right. Meanwhile, the game has been quantized – see The Quantum Monty Hall Problem by G. M. D'Ariano, R. D. Gill, M. Keyl, B. Kümmerer, H. Maassen, and R. F. Werner, Quant. Inf. Comp. **2** (2002), 355.