# Einführung in die Quantenoptik I

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**Problem 2.1** – About the size of a photon (5 points)

Quantum opticians are still discussing how strongly a single photon can be spatially localized. The central concept remains the same, however: a 'photon' with (mean) frequency $\nu$ has an energy $h\nu$ that is the space integral of the electromagnetic energy density $u = \frac{1}{2}\varepsilon_0 \mathbf{E}^2 + \frac{1}{2}\mu_0 \mathbf{H}^2$ over the region where the photon is 'localized'.

Take a photon in the visible range and estimate the strength of its electric and magnetic fields ($\mathbf{H}$ in $\mathrm{A/m}$ and $\mathbf{B}$ in $T$) for the following three scenarios: (1) photon completely delocalized over the volume of a Fabry-Pérot cavity (ask in the photonics lab about the typical size); (2) photon localized within the size of a wavelength (volume $\lambda^3$); (3) photon localized within the size of a hydrogen atom (Bohr radius $a_0 = 4\pi\varepsilon_0\hbar^2/me^2$) (Ole Keller, Aalborg, "Quantum Theory of Near Field Electrodynamics" 2011).

**Problem 2.2** – Field commutators (10 points)

In the lecture, we have seen the commutator between the fields operators $\mathbf{E}(x)$ and $\mathbf{H}(x)$:

$$[E_j(\mathbf{x}, t),\ H_k(\mathbf{x}', t)] = (\mathrm{const.})\mathrm{i}\hbar\epsilon_{jkl}\frac{\partial}{\partial x_l}\delta(\mathbf{x} - \mathbf{x}') \tag{2.1}$$

where the constant depends on the system of units ($\pm 1/(\varepsilon_0\mu_0)$ in SI units). All other commutators vanish. All fields in this problem have to be understood as operators.

(1) Fix the constant in Eq.(2.1) from the following 'correspondence principle': in the Heisenberg picture, the standard equation of motion for the magnetic field

$$\frac{\partial}{\partial t}\mathbf{H} = \frac{\mathrm{i}}{\hbar}[H,\ \mathbf{H}]$$

should reduce to the Faraday equation

$$\mu_0 \frac{\partial}{\partial t}\mathbf{H} = -\nabla \times \mathbf{E} \tag{2.2}$$

and the Hamiltonian $H$ is simply the space integral over the field energy

$$H = \int \mathrm{d}V \left(\frac{\varepsilon_0}{2}\mathbf{E}^2 + \frac{\mu_0}{2}\mathbf{H}^2\right) \tag{2.3}$$

**Hint**. Recall that the commutator acts on products similar to the product rule of differential calculus:

$$[H, AB] = [H, A]\, B + A\, [H, B]$$

The distribution 'derivative of $\delta$-function' in Eq.(2.1) is *defined* by the rules of partial integration. You can safely assume that there is no contribution from boundary terms.

(2) Consider the amplitude operators for smooth mode functions $\mathbf{f}$ and $\mathbf{h}$:

$$\mathcal{E} = \int \mathrm{d}^3x\, \mathbf{f}(\mathbf{x}) \cdot \mathbf{E}(\mathbf{x}), \qquad \mathcal{H} = \int \mathrm{d}^3x\, \mathbf{h}(\mathbf{x}) \cdot \mathbf{H}(\mathbf{x}) \tag{2.4}$$

and assume that the modes are related by $\omega\varepsilon_0 \mathbf{f} = \nabla \times \mathbf{h}$. Work out the commutator $[\mathcal{E}, \mathcal{H}]$ and conclude that orthogonal components of $\mathbf{E}$ and $\mathbf{H}$ at neighboring points cannot be measured simultaneously. Derive, as in the quantum mechanics I course, the uncertainty relation between the variances $(\Delta\mathcal{E})^2$, $(\Delta\mathcal{H})^2$. Use this to show that the electric and magnetic energies in these modes have a quantum uncertainty of the order of $\hbar\omega$.

**Problem 2.3** – Transverse $\delta$-function (5 points)

In the quantization of the e.m. field in free space, plane waves with transverse polarization vectors provide a convenient mode basis. (1) Argue that the 'completeness relation' of these modes is given by

$$\int \frac{\mathrm{d}^3k}{(2\pi)^3} \sum_\mu e_{\mu i}(\mathbf{k}) e_{\mu j}(\mathbf{k})\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = \delta_{ij}^{(T)}(\mathbf{x} - \mathbf{x}') \tag{2.5}$$

where $\mathbf{e}_\mu(\mathbf{k})$ are real, orthonormal polarization vectors perpendicular to $\mathbf{k}$. This integral defines the so-called transverse $\delta$-function. (2) Show first that the polarization sum gives

$$\sum_\mu e_{\mu i}(\mathbf{k}) e_{\mu j}(\mathbf{k}) = \delta_{ij} - \frac{k_i k_j}{\mathbf{k}^2} \tag{2.6}$$

and conclude that $\delta_{ij}^{(T)}(\mathbf{x} - \mathbf{x}')$ is symmetric in the indices $ij$. (3) To work out the $\mathbf{k}$-integral, start from the relation

$$\int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{k_i k_j}{\mathbf{k}^2}\, \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = -\frac{\partial^2}{\partial x_i \partial x_j} \int \frac{\mathrm{d}^3k}{(2\pi)^3} \frac{\mathrm{e}^{\mathrm{i}\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{\mathbf{k}^2} \tag{2.7}$$

and use the Fourier transform of the Coulomb ($1/|\mathbf{x} - \mathbf{x}'|$) potential. (4) Check the relative weight of the two terms by the following argument: a gradient field $\mathbf{F} = \nabla\phi$ is mapped to zero under the action of $\delta^{(T)}$:

$$\sum_j \int \mathrm{d}^3x'\, \delta_{ij}^{(T)}(\mathbf{x} - \mathbf{x}') F_j(\mathbf{x}') = 0 \tag{2.8}$$