Einführung in die Quantenoptik I

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Problem 7.1 – Canonical transformations (12 points)

In the lecture, we have seen several examples of unitary transformations, for example time evolution $U(t)$, displacement $D(\alpha)$, and beam splitter $S(\theta)$ operators.

(1) Show that all of these *preserve* commutation relations. This means: if a, $a^\dagger$ are bosonic operators with $[a, a^\dagger] = \mathbb{1}$ and the transformed operators are $A(t) = U^\dagger(t) a U(t)$ (similarly for $A^\dagger$), then we again have $[A(t), A^\dagger(t)] = \mathbb{1}$. Transformations with this property are called 'canonical transformations' and form a group.

(2) Generalise your proof to several bosonic operators, $a_1, \ldots a_n$. Show that a linear (affine) transformation with complex parameters M_{ij} and α_i,

$$A_i = \sum_j M_{ij} a_j + \alpha_i \,, \qquad A_i^\dagger = \sum_j M_{ij}^* a_j^\dagger + \alpha_i^* \,, \tag{7.1}$$

is canonical if and only if the $n \times n$ matrix M is unitary.

(3) In the lecture, we have often discussed differential equations involving transformations in exponential form, e.g., $S(\theta) = \exp(\mathrm{i}\theta \hat{J})$. Calculate the θ-derivative of the commutator $[A_i(\theta), A_j^\dagger(\theta)]$ and show that it is zero provided J is hermitian. Conclude that $S(\theta)$ is canonical.

(4) Work out the differential equations for the 'squeezed mode operators'

$$A(t) = S^\dagger(\xi, t) a S(\xi, t) \,, \qquad S(\xi, t) = \exp[t(\xi a^{\dagger 2} - \xi^* a^2)] \tag{7.2}$$

with a complex parameter ξ and explain why they 'mix' the annihilation and the creation operators. Check that the differential equations are solved by

$$A(t) = \mu(t) a + \nu(t) a^\dagger \,, \quad \mu(t) = \cosh(2t|\xi|) \,, \quad \nu(t) = \mathrm{e}^{\mathrm{i}\varphi} \sinh(2t|\xi|) \tag{7.3}$$

where φ is the phase angle of ξ.

Problem 7.2 – Homodyne measurement (8 points)

A beam splitter is the typical tool to 'mix' two fields. Take as an example a '50:50' beam splitter (amplitudes $\pm 1/\sqrt{2}$ in transmission and reflection) and consider in the input beam 1 a coherent state (amplitude α_1) and in input 2 a number state (photon number n_2). Calculate the average photon numbers $\langle N_1 \rangle$ and $\langle N_2 \rangle$ in the output beams with amplitudes

$$A_1 = \frac{a_1 - a_2}{\sqrt{2}}\,, \quad A_2 = \frac{a_2 + a_1}{\sqrt{2}} \tag{7.4}$$

and compute also the correlation functions

$$\langle A_1 A_2 \rangle - \langle A_1 \rangle \langle A_2 \rangle\,, \quad \langle N_1 N_2 \rangle - \langle N_1 \rangle \langle N_2 \rangle \tag{7.5}$$

In the experimental procedure called 'homodyne detection', the coherent state is very intense, $|\alpha_1|^2 \gg n_2$. Discuss your results in this limiting case and compare the signals to a direct measurement of the photon number in the input beam 2.