Chapter 3

Quantum theory of the laser

3.1 Lecture 12 Jan 15: laser cavity and rate equations

Spectral distribution of field intensity inside a Fabry-Pérot cavity: linewidth κ, 'photon lifetime' $1/\kappa$.

Reversible vacuum Rabi oscillations vs. exponential decay: role of 'shape' (or color) of spectral density.

Rate equations in the Scully-Lamb laser theory (see Sec.3.6): stimulated and spontaneous emission, photon loss.

Rate equations (see Problem 7.1) and stationary solution below threshold, see Section 3.6 on the theory by Scully & Lamb.

3.2 Introduction

Additional material not covered in detail.

What are the typical components of a laser[1]? Without going into details, we can identify two of them:

- some matter that amplifies light ("active medium");
- some device that traps the light around the space filled with the medium ("cavity")

[1] Acronym of "Light Amplifier by Stimulated Emission of Radiation"

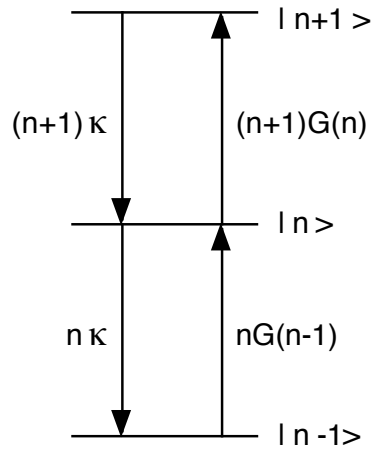


Figure 3.1: Transitions between photon number states.

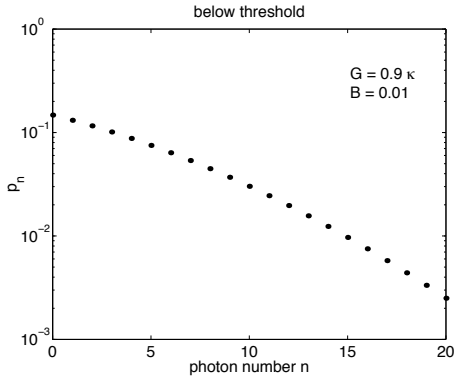


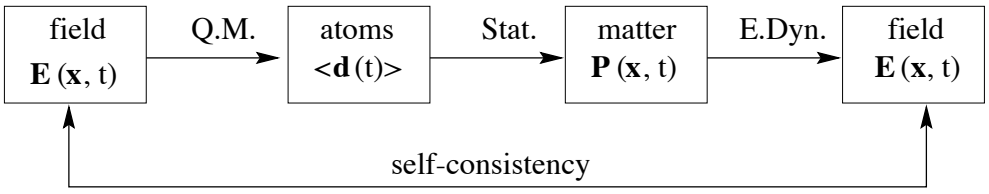
Figure 3.2: Photon statistics of a laser in steady state. Below threshold $G \equiv G_0 < \kappa$, logarithmic scale.

In order to obtain an amplifying medium, one has to "pump" energy into it. The gain medium is thus a converter between the pump energy and the light emission. Quite often, the conversion efficiency is low, with values in the range 10–50% being considered "large".

The feedback mechanism is needed because the light would otherwise escape from the medium. An optical cavity like a Fabry-Pérot resonator (two mirrors) does this job because the light can travel back and forth between the mirrors a large number of times.

General ideas

Sargent and Scully (1972) propose the following diagram that relates the different theories needed to describe a laser.



The electromagnetic field drives microscopic dipoles in the laser medium. This was the topic of the previous term. A statistical description gives, implicit in the density matrix approach we followed, links the dipoles to the macroscopic polarization of the medium [see Eq. (3.14) below]. The polarization enters the Maxwell equations for the electromagnetic field as a source and generates the field. In the end, a self-consistent description is required: the fields at the left and right end should coincide. The condition of self-consistency allows to derive the following important quantities:

- the laser threshold,
- the laser intensity in steady state,
- the laser frequency.

This can be achieve even when one treats the field treated classically. For example, a “classical” or “coherent” field appeared via the Rabi frequency in the Bloch equations of the previous term. This approach is called “semi-classical laser theory”. Note that nowhere in this approach does the word “photon” appear (if one is serious).

When a quantum-mechanical description is adopted, the photon finally comes into play and one may also derive

- the photon number probabability distribution (“photon statistics”),
- the intensity fluctuations and correlations of laser light,
- the phase fluctuations (related to the laser linewidth).

We shall illustrate this quantum theory by a calculation of the photon statistics and the laser linewidth.

We focus in these elementary considerations on a homogeneously distributed medium in the cavity made up from identical two-level systems ("homogeneous broadening"). Please refer to the experimental physics lectures for the discussion of "inhomogeneous" frequency broadening due to, for example, the atomic motion and other features.

3.3 Cavity field

(from SS 2014)

Reduction of the full electric field operator to a single mode:

$$\mathbf{E}(\mathbf{r},t) = \mathcal{E}_{1\mathrm{ph}}\hat{a}(t)\mathbf{f}(\mathbf{r}) + \mathrm{h.c.} \tag{3.1}$$

where the 'mode function' $\mathbf{f}(\mathbf{r})$ describes the spatial pattern of the cavity mode. It is the result of classical electrodynamics. A simplified form could be

$$\mathbf{f}(\mathbf{r}) \approx \mathbf{e}\mathcal{N}\frac{\sin(kz)}{\sqrt{Lw^2}}\mathrm{e}^{-(x^2+y^2)/w^2} \tag{3.2}$$

where $\mathbf{e}$ is a polarization vector and $\mathcal{N}$ is a numerical factor determined by the normalization integral

$$1 \stackrel{!}{=} \int_V \mathrm{d}V\,|\mathbf{f}(\mathbf{r})|^2 \tag{3.3}$$

The volume V of the cavity has length L in the z-direction. Mode functions like Eq.(3.2) appear naturally in the so-called paraxial approximation where one assumes that the mode is concentrated around the cavity axis (the z-axis). The 'waist' w gives the size transverse to the axis (in the xy-plane), in the paraxial approximation, we have $kw \gg 1$. The mode function $\mathbf{f}(\mathbf{r})$ solves a wave equation for a fixed frequency (vector Helmholtz equation); we call this the 'cavity frequency' ω_c in the following.

The electric field $\mathbf{E}(\mathbf{r},t)$ becomes an operator through the time-dependent amplitude $\hat{a}(t)$ and its hermitean conjugate $\hat{a}^\dagger(t)$. These operators annihilate ($\hat{a}(t)$) and create ($\hat{a}^\dagger(t)$) photons in the cavity – they act as ladder operators in the same way as for the harmonic oscillator of elementary quantum mechanics. Indeed, the energy spectrum of the cavity is given by the Hamiltonian operator

$$\hat{H}_c = \hbar\omega_c\hat{a}^\dagger(t)\hat{a}(t) \tag{3.4}$$

The basic relation between the creation and annihilation operators is the commutator

$$\left[\hat{a}(t),\, \hat{a}^\dagger(t)\right] = 1 \tag{3.5}$$

where 1 denotes the unit operator. This relation holds only if both operators are evaluated at the same time. Remember that from this, one gets that the eigenvalues of $\hat{a}^\dagger(t)\hat{a}(t)$ are $0, 1, 2\ldots$: this operator is called the

$$\text{number operator } \hat{n} = \hat{a}^\dagger(t)\hat{a}(t) \tag{3.6}$$

and that the energy eigenvalues are

$$E_n = \hbar\omega_c n \tag{3.7}$$

Up to the choice of the ground state energy, the excited states of the cavity can therefore be understood as carrying a number $n = 1, 2\ldots$ of energy quanta $\hbar\omega_c$, one per photon.

Free evolution of the cavity field. In the Schrödinger picture, in the number state basis $|n\rangle$

$$|\psi(0)\rangle = \sum_{n=0}^{\infty} c_n|n\rangle \quad \rightarrow \quad |\psi(t)\rangle = \sum_{n=0}^{\infty} c_n \mathrm{e}^{-\mathrm{i}n\omega_c t}|n\rangle \tag{3.8}$$

For the density operator $\rho(t) = |\psi(t)\rangle\langle\psi(t)|$

$$\mathrm{i}\hbar\partial_t\rho = [H_c,\, \rho] \quad \rightarrow \quad \rho_{nm}(t) = \rho_{nm}(0)\, \mathrm{e}^{-\mathrm{i}(n-m)\omega_c t} \tag{3.9}$$

where the matrix elements are denoted

$$\rho_{nm} = \langle n|\rho|m\rangle \tag{3.10}$$

Observe that the *populations* ρ_{nn} remain constant, while the *coherences* ρ_{nm} with $n \neq m$ oscillate at multiples of the cavity frequency.

In particular, for the expectation value of the cavity field, we need to compute (in the Schrödinger picture)

$$\begin{aligned}
\langle\hat{a}(t)\rangle &= \langle\psi(t)|a|\psi(t)\rangle = \mathrm{tr}[a\rho(t)] \\
&= \sum_n \langle n|a\rho(t)|n\rangle = \sum_{n,m}\langle n|a|m\rangle\langle m|\rho(t)|n\rangle \\
&= \sum_{n,m}\delta_{n-1,m}\sqrt{m}\rho_{mn}(t) = \sum_m \sqrt{m}\rho_{m,m-1}(t)
\end{aligned} \tag{3.11}$$

See how the average field involves this sum over coherences. Inserting the free cavity evolution, we find

$$\langle\hat{a}(t)\rangle = \sum_m \sqrt{m}\rho_{m,m-1}(0)\mathrm{e}^{-\mathrm{i}\omega_c t} \tag{3.12}$$

where the exponential contains just once the cavity frequency.

Exercise. In the Heisenberg picture, $\hat{a}(t) = \hat{a}(0)\,\mathrm{e}^{-\mathrm{i}\omega_c t}$ which gives the same result, from a slightly shorter calculation. Similarly, $\hat{n}(t) = \hat{n}$ for a free cavity.

Note. We now have to add to the description the active medium and the loss from the cavity (to get at least some laser beam out of the device).

3.4 Active medium

The "active medium" cavity inside the cavity that provides the polarization consists, in many cases, of a large number of atoms or molecules. These atoms are prepared in the excited state by some process that feeds energy into them ("pumping mechanism"), and then wait to release their energy in the form of photons into the cavity field. A two-level approximation for the atoms is a simple way to account for the sharp, nearly monochromatic emission spectrum of the laser. We could have used as well a harmonic oscillator[2], however, this does not reproduce some basic features of the laser like gain saturation. The quantum theory for the atom-light interaction gives us an expression for the "microscopic", average electric dipole $\langle \mathbf{d}(t) \rangle$. The polarization field is then simply the number density of these dipoles

$$\mathbf{P}(\mathbf{x}, t) = N(\mathbf{x}) \langle \mathbf{d}(t) \rangle. \tag{3.13}$$

In general, the density $N(\mathbf{x})$ is position dependent. In fact, also the induced dipole is because it involves the light field at the position $\mathbf{x}$.

The complex, slowly varying polarization field can be connected to the coherences of the density matrix in the rotating frame:

$$\begin{aligned} \mathbf{P}(\mathbf{x}, t) &= N(\mathbf{x}) \left[\langle \mathbf{d}^{(+)}(t) \rangle + \text{c.c.} \right] \\ &= N(\mathbf{x}) \mathbf{d} \left[\rho_{\text{eg}}(t) \mathrm{e}^{-\mathrm{i}\omega_L t} + \text{c.c.} \right] \end{aligned} \tag{3.14}$$

where $\mathbf{d}^{(+)}(t)$ is the positive frequency part of the atomic dipole operator, $\mathbf{d}$ is the fixed vector of dipole matrix elements and $\rho_{\text{eg}}(t)$ is the off-diagonal element ("optical coherence") of the atomic density matrix in the frame rotating at ω_L. A formula like (3.14) assumes that all dipoles in the medium are driven by a similar field and do not interact with each other. This is a first starting point and leaves place for more elaborate theories, of course. We assume in particular that

[2] This is a good model for antennas emitting at radio frequencies.

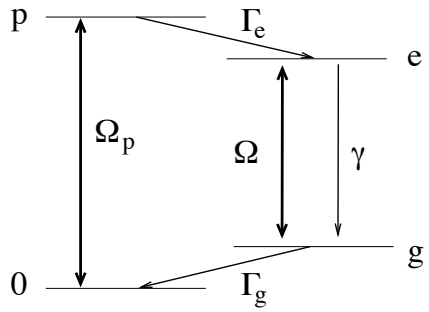


Figure 3.3: Four-level model to describe incoherent population pumping of the upper state e of the lasing transition e $\leftrightarrow$ g.

the resonance frequency is the same for all microscopic dipoles. This is not true for atoms in a (thermal) gas where the Doppler effect leads to a distribution of the resonance frequencies ("inhomogeneous broadening").

Polarization and saturation

We now have to find a way to determine the dipole moment of the two-level atoms. Recall that its imaginary part is essential for light amplification. We shall see that it depends on the atomic inversion (population difference between upper and lower state). To this end, we use the optical Bloch equations for the atomic density matrix, with some modifications by adding additional energy levels. This model also provides a better understanding of the "pumping" mechanism, beyond some phenomenological rate equations. The modified two-level system is for example a four-level atom with fast relaxation in the two upper and two lower states. A simple model with four states as shown in figure 3.3 is outlined in the exercises. In this limit, the optical Bloch equations can be simplified, and one gets a justification for the often-used rate equations.

The next task is to compute the optical coherence $\rho_{\mathrm{eg}}(t)$ from the optical Bloch equations. Let us write down these equations for the two levels e and g involved in the laser transition. The rate equations for the populations ρ_{ee} and ρ_{gg} involve a pumping rate $\lambda_{\mathrm{e}} = \Gamma_{\mathrm{e}}\rho_{\mathrm{pp}}$ into the upper state (via rapid decay from the pumped state p), the spontaneous decay rate γ and a decay rate γ_{g} for the lower state. Including the Rabi frequency $\Omega = -(2/\hbar)\mathbf{d}\cdot\mathbf{E}$ for the laser field, as you have

learned in the previous semester, this gives

$$\dot{\rho}_{\rm ee} = \lambda_{\rm e} - \gamma\rho_{\rm ee} + {\rm i}\frac{\Omega}{2}\left(\rho_{\rm eg} - \rho_{\rm ge}\right), \tag{3.15}$$

$$\dot{\rho}_{\rm gg} = \gamma\rho_{\rm ee} - \gamma_{\rm g}\rho_{\rm gg} - {\rm i}\frac{\Omega}{2}\left(\rho_{\rm eg} - \rho_{\rm ge}\right). \tag{3.16}$$

Notice that the first equation gives an increase of the excited state population when the coherence $\rho_{\rm eg}$ has a positive imaginary part (recall that Ω is actually negative...). This is in agreement with the damping of the field energy in the wave equation derived before.

The last Bloch equation is for the coherence itself. The population decay rates γ and $\gamma_{\rm g}$ lead to a decoherence rate $\Gamma = \frac{1}{2}(\gamma + \gamma_{\rm g})$ as you have seen in the derivation of the Bloch equations. In the frame rotating at the laser frequency ω_L, the laser field detuning is $\Delta = \omega_L - \omega_{\rm eg}$, and we get

$$\dot{\rho}_{\rm eg} = {\rm i}\Delta\rho_{\rm eg} - \Gamma\rho_{\rm eg} + {\rm i}\frac{\Omega}{2}\left(\rho_{\rm ee} - \rho_{\rm gg}\right). \tag{3.17}$$

From this equation we learn that the optical dipole is created by the population difference (inversion) $\rho_{\rm ee} - \rho_{\rm gg}$. As discussed in the exercises, this equation can be solved approximately in the limit that the decay rate Γ is the largest time constant around (this solution also corresponds to the stationary state):

$$\rho_{\rm eg} = -\frac{\Omega/2}{\Delta + {\rm i}\Gamma}\left(\rho_{\rm ee} - \rho_{\rm gg}\right). \tag{3.18}$$

In particular, the imaginary part of the optical coherence is (we assume as usual a real Rabi frequency)

$${\rm Im}\,\rho_{\rm eg} = \frac{\Gamma\Omega/2}{\Delta^2 + \Gamma^2}\left(\rho_{\rm ee} - \rho_{\rm gg}\right).$$

Note that this expression is negative when the two-level system is inverted (upper level population $\rho_{\rm ee} > \rho_{\rm g}$), using again that actually $\Omega < 0$. This means that the medium amplifies the light via stimulated emission.

Solving also the other Bloch equations in the stationary state, we can compute the inversion

$$\rho_{\rm ee} - \rho_{\rm gg} = \lambda_{\rm e}\left(\frac{1}{\gamma} - \frac{1}{\gamma_{\rm g}}\right)\frac{\Delta^2 + \Gamma^2}{\Delta^2 + \Gamma^2 + (\Gamma/\gamma)\Omega^2/2}. \tag{3.19}$$

The system is inverted when the lifetime $1/\gamma$ of the upper state exceeds the lifetime $1/\gamma_{\rm g}$ of the lower state, which is perfectly reasonable.

The end result of the calculation is the following expression for the polarization field. We quote only the amplitude of the positive frequency component and assume that dipole moment and electric field are collinear:

$$\mathbf{P}(\mathbf{x}) = N(\mathbf{x})(D^2/\hbar)\mathbf{e}E(\mathbf{x})\frac{\lambda_\mathrm{e}\,(1/\gamma - 1/\gamma_\mathrm{g})\,(\Delta - \mathrm{i}\Gamma)}{\Delta^2 + \Gamma^2 + 2(\Gamma D^2/\gamma\hbar^2)|E(\mathbf{x})|^2} \tag{3.20}$$

$$\text{or} \qquad \mathbf{P}(\mathbf{x}) = \frac{\varepsilon_0\chi\mathbf{E}(\mathbf{x})}{1 + B|E(\mathbf{x})|^2}. \tag{3.21}$$

In the last line, we have introduced the (linear) susceptibility χ of the laser medium and a coefficient B that takes into account the nonlinear response.

The most important result is that the imaginary part of the polarization is negative (*amplification* of the field) when the two-level system is inverted. The four-level scheme shown in fig. 3.3 is just one possibility to achieve inversion by a suitable pumping scheme. Refer to the experimental physics lectures for other, perhaps more efficient, pumping mechanisms.

The coefficient B in (3.21) describes the *saturation* of the medium: for very large laser intensity $|E|^2$, the induced polarization decreases proportional to $1/|E|$ instead of increasing. The physics behind saturation is characteristic for the two-level system: when the laser field gets extremely strong, the inversion vanishes (see Eq. (3.19)). We have already seen this behaviour when we considered Rabi oscillations with weak damping: the two-level system gets always re-excited by the laser and is finally with equal probability in the upper and lower states. For a harmonic oscillator, there is no saturation since arbitrarily high lying states can be populated.

3.5 Wave equation for the field

(not covered in WS 15/16)
We start with a reminder of the electrodynamics in a material with a given polarization. Let us recall that the polarization field enters the following Maxwell equation:

$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \mathbf{j} + \frac{\partial}{\partial t}\left(\varepsilon_0\mathbf{E} + \mathbf{P}\right)$$

where it gives the "bound" part of the current density. We put in the following the "free" current density $\mathbf{j} = \mathbf{0}$ because we assume that the active material is globally neutral and the light only induces dipoles in it. Combining with the Faraday induction equation,

$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}$, one gets the wave equation for the electric field where the polarization enters as a source term:

$$\nabla \times \nabla \times \mathbf{E} + \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = -\mu_0 \frac{\partial^2}{\partial t^2}\mathbf{P}. \tag{3.22}$$

We now make the approximation that in the cavity, a single mode is sufficient to capture the field dynamics. You have seen that one can then write for the field operator

$$\mathbf{E}(\mathbf{x}, t) = \mathcal{E}_{1\mathrm{ph}} \left[\mathbf{f}(\mathbf{x})a(t) + \mathbf{f}^*(\mathbf{x})a^\dagger(t)\right] \tag{3.23}$$

where $\mathcal{E}_{1\mathrm{ph}} = (\hbar\omega_L/2\varepsilon_0)^{1/2}$ is the "one-photon field amplitude" (up to a factor $1/\sqrt{V}$ where V is the 'mode volume' that appears in $\mathbf{f}$). $a(t)$ is the annihilation operator for the mode (time-dependent in the Heisenberg picture), and $\mathbf{f}(\mathbf{x})$ is the spatial mode function. It solves the homogeneous equation

$$\nabla \times \nabla \times \mathbf{f} - \frac{\omega_c^2}{c^2}\mathbf{f} = \mathbf{0} \tag{3.24}$$

as you remember from the field quantization procedure. Here ω_c is one of the (empty) cavity resonance frequencies. The mode function is normalized such that the integral of its square over the cavity volume V gives 1, Eq.(3.3). There are lasers where propagating modes are a suitable description. In the photonics lectures, other cavity modes, including their transverse behaviour (perpendicular to the cavity axis) are introduced. For the semiclassical theory we develop here first, the product $\mathcal{E}_{1\mathrm{ph}}a(t) = E(t)$ gives the (positive frequency) field amplitude. Its absolute square corresponds to the intensity, with $a^*(t)a(t)$ giving the "photon number" (although this is not required to be an integer in the semiclassical theory).

We now project the wave equation (3.22) onto the field mode $\mathbf{f}(\mathbf{x})$. The term involving $\mathbf{f}^*(\mathbf{x})a^\dagger(t)$ does not contribute when a propagating mode is used. We also make the approximation of slowly varying amplitudes for $E(t)$ that oscillates essentially at the frequency ω_L. The polarization field as well, $\mathbf{P}(\mathbf{x}, t) = \mathbf{P}(\mathbf{x})\,\mathrm{e}^{-\mathrm{i}\omega_L t} + \mathrm{c.c.}$ This means that the time derivative of $\mathbf{P}(\mathbf{x})$ is much smaller than $\omega_L\mathbf{P}(\mathbf{x})$. (With this approximation and standing wave modes, the term $\mathbf{f}^*(\mathbf{x})a^\dagger(t)$ drops out now.) We get

$$\dot{E} = -\mathrm{i}(\omega_L - \omega_c)E - \frac{\kappa}{2}E + \mathrm{i}\frac{\omega_L}{2\varepsilon_0}\int_V \mathrm{d}^3x\, \mathbf{f}^*(\mathbf{x}) \cdot \mathbf{P}(\mathbf{x}), \tag{3.25}$$

where $\omega_L - \omega_c$ is the frequency detuning with respect to the cavity resonance and V the cavity volume. We have introduced the phenomenological decay rate κ for the energy of the cavity field. The quality factor of the cavity (often known experimentally) is given by $Q = \omega_c/\kappa$. Notice that the spatial integral is the overlap of the polarization field with the cavity mode. It is easy to see from this equation that the *real* part of the polarization

$\mathbf{P}$ determines a frequency shift of the laser (with respect to the cavity frequency), and that its *imaginary* part changes the energy $\propto |E|^2$ of the field. In particular, if $\mathrm{Im}\,\mathbf{P}$ is negative, the field energy increases (emission). We thus anticipate to find the absorption and emission of the medium in the imaginary part of the polarization.

We now want an equation for the intensity $I(t) = |E(t)|^2$ (restoring a slow time-dependence) in the mode $\mathbf{f}(\mathbf{x})$. To this end, we work out the spatial overlap in Eq. (3.25) with a standing wave mode $\mathbf{f}(\mathbf{x}) \sim \mathbf{e}\, L^{-1/2} \sin kz$:

$$\dot{E} = -\mathrm{i}(\omega_L - \omega_c)E - \frac{\kappa}{2}E + \mathrm{i}\frac{\omega_L \chi}{2}E(t) \int \frac{\mathrm{d}z}{L} \frac{2\sin^2(kz)}{1 + 2B|E(t)|^2 \sin^2(kz)}$$

Here, L is the cavity length. The difficulty is the sine function in the denominator. In the exercises, you are asked to compute this integral analytically. Here, we adopt an approximate treatment that is also often used in the literature and assume that the saturation is weak. The denominator can then be expanded, and to first order in B, we get

$$\int \frac{\mathrm{d}z}{L} 2\sin^2(kz) \left[1 - 2B|E(t)|^2 \sin^2(kz)\right] = 1 - \frac{3B}{2}|E(t)|^2$$

As an exercise, you can estimate the dimensionless quantity $B|E(t)|^2$ for typical parameter values. This result is often "resummed" to make the saturation effect more clear:

$$1 - \frac{3B}{2}|E(t)|^2 \approx \frac{1}{1 + \frac{3B}{2}|E(t)|^2}$$

This procedure may seem strange, but reproduces quite well the exact result, as shown in figure 3.4.

The equation of motion for the intensity is now given by (for small gain saturation; check that B is real)

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 2\,\mathrm{Re}\left(E^* \frac{\mathrm{d}E}{\mathrm{d}t}\right) = -\kappa I(t) - \omega_L(\mathrm{Im}\,\chi) I(t) \left[1 - \frac{3BI(t)}{2}\right] \tag{3.26}$$

This form suggests the introduction of an amplification rate ("gain") $G = -\omega_L(\mathrm{Im}\,\chi)$. Using also the conventional notation $\beta = 3GB/2$ (a rate per intensity) for the saturation coefficient, we obtain

$$\frac{\mathrm{d}I}{\mathrm{d}t} = (G - \kappa)I - \beta I^2 \tag{3.27}$$

as the fundamental equation of motion for the laser intensity in the semiclassical theory.

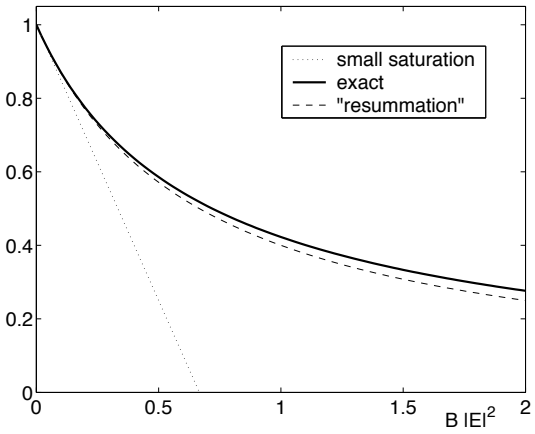
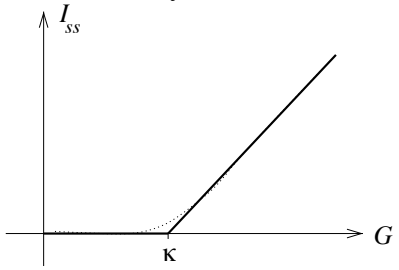


Figure 3.4: Different approximations for the gain saturation factor, integrated over the cavity mode.

Exercise. Laser threshold and dependence of steady-state intensity $I_{\rm ss}$ on the gain. Dotted: result of quantum theory: the threshold becomes smooth.



3.6 Scully-Lamb master equation

In this section, we outline a theory of the laser that starts from a quantum description of the cavity field. We still use for simplicity the single mode approximation — the basic observables are hence the annihilation and creation operators $a, a^\dagger$ for the field mode.

The laser is an *open quantum system* because energy is continuously fed into and removed from the cavity mode. We therefore have to use a density matrix description, as we did in the first part for a two-level atom. What are the "reservoirs" that the field mode is coupled to? First of all, the mode continuum outside the cavity: part of the cavity losses show up here (and permit to observe the laser dynamics). But in general, losses also occur in the material that

makes up the cavity: mirrors and optical elements. We do not develop in this semester's course a detailed quantum theory of lossy optical elements (see earlier versions). Finally, the laser medium is also a reservoir of energy that may flow into the field mode — or not when the medium spontaneously emits photons into other modes.

In this section, we recall the master equation description for linear cavity loss and motivate the corresponding model for the gain medium. We shall derive a rate equation for the probabilities of finding n photons in the laser mode whose stationary solution gives the photon statistics. Finally, a sketch is given of the Schawlow-Townes limit for the laser linewidth.

Cavity damping

The density operator for the cavity field, $\rho(t)$, acts on the Hilbert space for the harmonic oscillator associated with the field mode. Taking the trace, we find the quantum expectation values of the quantities of interest. The average electric field, for example, is given by (we only write the positive frequency part)

$$\langle \mathbf{E}(\mathbf{x},t)\rangle = \mathbf{f}(\mathbf{x})\mathcal{E}_{1\mathrm{ph}}\langle a(t)\rangle = \mathbf{f}(\mathbf{x})\mathcal{E}_{1\mathrm{ph}}\,\mathrm{tr}\,[a\,\rho(t)]$$

The trace can be performed in any basis, using photon number states or coherent states, for example. In the absence of any interaction, the Heisenberg operator a evolves freely at the frequency ω_c of the cavity. (We suppose for simplicity that this coincides with the laser frequency.)

(Lindblad master equations not covered in SS 2015)
We have seen in chap. **??** that the master equation to describe linear damping can be written in terms of Lindblad or jump operators. The one we need for cavity damping is given by $L = \sqrt{\kappa}\,a$, leading to

$$\left.\frac{\mathrm{d}\rho}{\mathrm{d}t}\right|_\mathrm{damp} = \kappa\, a\rho a^\dagger - \frac{\kappa}{2}\left\{a^\dagger a,\,\rho\right\}. \tag{3.28}$$

It is easy to check that the rate κ has the same meaning as in the semiclassical theory: it gives the (exponential) decay of the field's photon number if no other dynamics is present.

As an exercise, you may want to derive the rate equations for the diagonal elements $p_n(t) = \langle n|\rho(t)|n\rangle$ of the density matrix (these form the 'photon statistics'). The master equation (3.28) gives a transition rate between the photon number states $|n\rangle$ and $|n-1\rangle$

that is given by $n\kappa$, proportional to the number of photons that are presently in the cavity mode. One is tempted to interpret this as "each photon decides independently to leave the cavity." The final state is the vacuum state with zero photons — this is related to the implicit assumption that the reservoir is at zero temperature. It is a reasonable approximation at optical frequencies and room temperature.

Gain

In the previous semester, we used a model with a driven field mode where a "pump" generates a coherent state. We cannot use this model any longer because the laser medium does not provide, a priori, a fixed phase reference for the field it generates. At least the spontaneous emission of the pumped two-level atoms is "incoherent" (no fixed phase).

A suitable model for cavity gain in the linear regime is given by the jump operator $L_2 = \sqrt{G}\, a^\dagger$ and the master equation

$$\left.\frac{\mathrm{d}\rho}{\mathrm{d}t}\right|_{\text{gain}} = G\, a^\dagger \rho a - \frac{G}{2}\left\{aa^\dagger,\, \rho\right\}, \tag{3.29}$$

where G is the gain rate coefficient and, up to the exchange of a and $a^\dagger$, no sign changes occur.

What about gain saturation? It is included in this theory if we allow G to depend on the instantaneous intensity of the cavity mode. The gain thus depends on the photon number, $G = G(n)$. By analogy to the semiclassical gain, one can use the model

$$G(a^\dagger a) = \frac{G_0}{1 + B a^\dagger a} \tag{3.30}$$

where $1/B$ plays the role of a saturation photon number. This actually leads to complications in the construction of suitable Lindblad operators. (How to take the square root here?) There are examples in textbooks that work with non-Lindblad forms and that need to be corrected afterwards to avoid unphysical results like negative probabilities. A discussion of this problem for a simple model ('micro maser') is sketched in Sec.3.8.

Photon statistics

To start our analysis, let us compute the rate equations for the populations $p_n \equiv \rho_{nn}$ of finding n photons in the cavity mode. The sum of damping and gain gives the master equation (we are cheating with the photon number in $G(a^\dagger a)$, as the trace is no longer conserved)

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\mathrm{i}\omega_\mathrm{c}\left[a^\dagger a,\, \rho\right] - \frac{\kappa}{2}\left\{a^\dagger a,\, \rho\right\} + \kappa\, a\rho a^\dagger$$

$$-\frac{G(a^\dagger a)}{2}\left\{aa^\dagger,\,\rho\right\}+G(a^\dagger a)\,a^\dagger\rho a. \tag{3.31}$$

Taking the expectation value in the state $|n\rangle$ of the master equation (3.31), we get (cheating again with the photon number in $G(n)$)

$$\frac{\mathrm{d}p_n}{\mathrm{d}t}=-n\kappa\,p_n+(n+1)\kappa\,p_{n+1}-(n+1)G(n)\,p_n+nG(n-1)\,p_{n-1} \tag{3.32}$$

From the terms with a negative sign, we see that transitions leave the state $|n\rangle$ with rates $n\kappa$ and $(n+1)G(n)$. Looking at the rate equation for the state $|n-1\rangle$, we see that population from state $|n\rangle$ arrives at a rate $n\kappa$. We have thus identified a first process: the cavity field loses one photon at the rate $n\kappa$. This is the expected loss process. But there is also a transition from $|n\rangle$ to $|n+1\rangle$, occurring at a rate $(n+1)G(n)$. This is both spontaneous ("+1") and stimulated emission ("n") from the laser medium. Note that the present theory requires G to be positive (inverted medium) because transition rates are positive. The dependence of $G(n)$ on the photon number again models the gain saturation, as was the case in the semiclassical theory.

The transitions we have found are summarized in figure 3.5. We can now

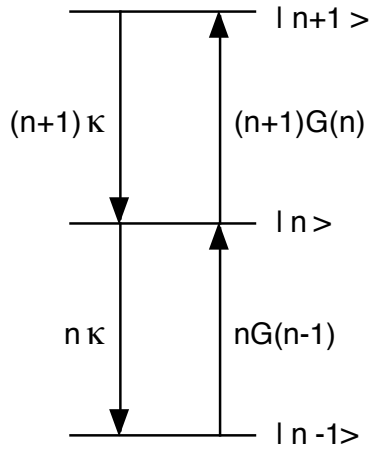


Figure 3.5: Transitions between photon number states.

determine the stationary state of the laser. The probabilities p_n and p_{n+1}, say, then do not change with time, and therefore the probability current for the loss process $|n+1\rangle\rightarrow|n\rangle$ must be equal to the current for the emission process $|n\rangle\rightarrow|n+1\rangle$:

$$(n+1)\kappa p_{n+1}=(n+1)G(n)p_n \tag{3.33}$$

This condition is called '*detailed balance*' and plays an important role in statistical physics.

For the pair of levels $|n\rangle$ and $|n-1\rangle$, we get $n\kappa p_n = nG(n-1)p_{n-1}$ from the rate equation (3.32) which can also be found by shifting the label n in Eq.(3.33). We may ask whether for the transition $|0\rangle \leftrightarrow |1\rangle$, there is saturation or not: can the field of a single photon saturate the laser medium? The answer cannot come from the semiclassical model we started with because that for that model the single-photon field is vanishingly small. But there are quantum models for a laser medium that show this saturation effect, under special conditions (typically, a high-quality laser cavity with a small volume is needed where the single-photon field is large enough).

We observe that the dynamic equilibrium between loss and gain processes gives a recurrence relation for the photon number probabilities in the stationary state. It is easily solved with the saturation model (3.30) to give

$$p_{n+1} = \frac{G(n)}{\kappa} p_n \Rightarrow p_n = \mathcal{N} \prod_{m=0}^{n-1} \frac{G(m)}{\kappa} = \mathcal{N} \left(\frac{G_0}{\kappa}\right)^n \prod_{m=0}^{n-1} \frac{1}{1+Bm}, \tag{3.34}$$

where $\mathcal{N}$ is a normalization constant. Below threshold, $G_0 < \kappa$, each of the ratios $G(n)/\kappa$ is smaller than unity, and the most probable state is the vacuum — perfectly reasonable because the laser intensity is damped away. Above threshold and for weak saturation, $G(n)/\kappa \approx G_0/\kappa > 1$, and photon numbers larger than zero are favoured. The maximum of the distribution is reached at a photon number $n_{\max}$ where $G(n_{\max})/\kappa = 1$. This equation can be solved to give

$$n_{\max} = \frac{G_0 - \kappa}{\kappa B}$$

which looks very similar to the steady state intensity of the semiclassical theory.

The photon statistics (3.34) is plotted in figure 3.6 for a laser below and above threshold. Note that below threshold, we do not have a thermal state (the probability is not an exponential $\propto \mathrm{e}^{-\beta n\hbar\omega_c}$), and that above threshold, the width of the number distribution is larger than for a coherent state with the same most probable photon number.

Exercise. Use the following representation of the product in (3.34)

$$\prod_{m=0}^{n-1} \frac{1}{1+Bm} = B^{-n} \frac{\Gamma(1/B)}{\Gamma(1/B+n)}$$

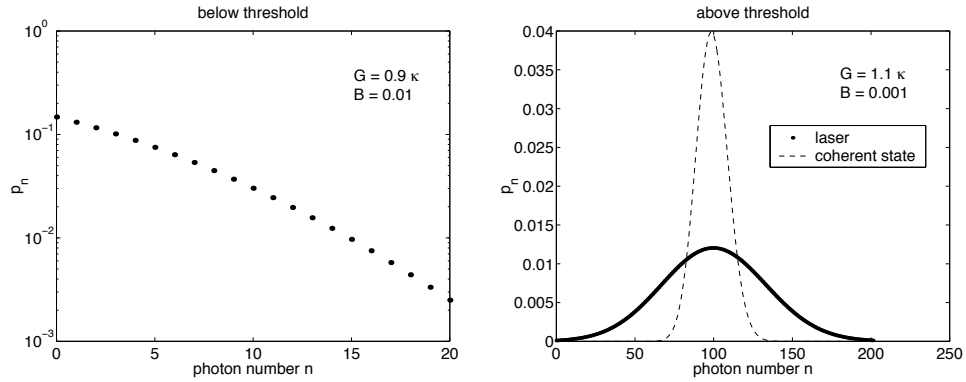


Figure 3.6: Photon statistics of a laser in steady state. Left: below threshold $G \equiv G_0 < \kappa$, right: above threshold.

where $\Gamma(\cdot)$ is now the gamma function to discuss the shape of the photon statistics. Using the Stirling formula for large values of n and $1/B$, show that p_n has the form of a truncated gaussian distribution and compute its width. You will find that the width approaches that of a coherent state (also known as Poisson statistics)

$$\Delta n^2 \to n_{\max}$$

when the laser is operating far above threshold. (This is difficult to achieve in practice, however.)

In Fig.3.7 below, we show the photon statistics (right) and the so-called Mandel parameter

$$Q = \frac{(\Delta n)^2}{\langle \hat{n} \rangle} \tag{3.35}$$

for a specific laser model that qualitatively behaves similar to that of Scully & Lamb. (More details in Sec.3.8.) The Mandel parameter takes the value $Q = 1$ if the photon statistics is Poissonian – this is reached well above the threshold. Below threshold, this is also true, although not very interesting since the total output intensity $\sim \langle \hat{n} \rangle$ is small. The peak of Q near the threshold is a manifestation of so-called 'critical fluctuations' – the system explores a large range of values around the mean because it is switching between two qualitatively different 'phases' (adopting the language of a phase transition).

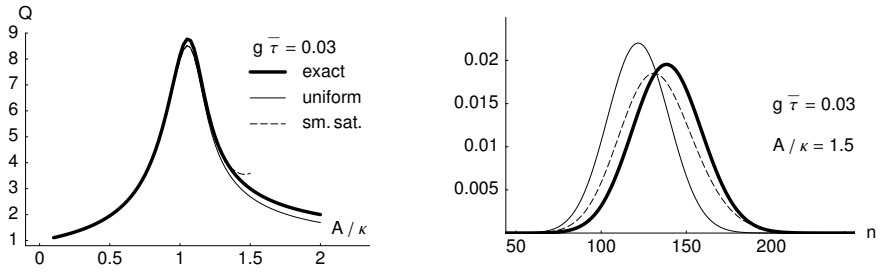


Figure 3.7: (*left*) Mandel parameter for a micromaser model, calculated with different approximations to the master equation (lines, 'uniform' = uniform approximation, 'sm. sat' = expansion for small saturation). The parameter $g\bar{\tau}$ qualitatively describes saturation, and corresponds to a parameter $B = (g\bar{\tau})^2$ in the Scully & Lamb model. The linear gain rate G_0 is denoted A in the plot, and κ is the loss rate of the laser cavity.
(*right*) Photon statistics above threshold, calculated with the same approximations. The distribution functions p_n are represented by continuous lines.

3.7 Laser linewidth and phase diffusion

Idea: phase diffusion

(In SS 2015, material on correlations appears later.)
The spectrum of the laser is related to the autocorrelation function of the laser mode operator a:

$$S(\omega) = \lim_{t\to\infty} \int \mathrm{d}\tau \, \mathrm{e}^{-\mathrm{i}\omega\tau} \langle a^\dagger(t+\tau) a(t) \rangle \tag{3.36}$$

where the average is taken in the stationary state that is reached at large times. We shall denote this state ρ_{ss} or by the subscript $\langle \ldots \rangle_{\mathrm{ss}}$. If the mode were evolving freely at the laser frequency, $a(t+\tau) = a(t)\,\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{L}} t}$, then the spectrum would be a δ-peak, $S(\omega) = \langle a^\dagger a \rangle_{\mathrm{ss}} 2\pi\delta(\omega - \omega_{\mathrm{L}})$ and proportional to the mean photon number. This is no longer true when we take into account that the gain mechanism also involves "spontaneous emission" where atoms of the pumping medium emit a spontaneous photon (not stimulated, hence no phase relation with the existing laser field). This gives the laser field a "fluctuating amplitude" that we have to characterize.

The main idea of Schawlow & Townes is that the laser field is essentially subject to phase fluctuation, but not to intensity fluctuations. In a semiclassical description, we thus have a mode amplitude $a(t) = \sqrt{\bar{n}_{\mathrm{ss}}}\,\mathrm{e}^{\mathrm{i}\phi(t) - \mathrm{i}\omega_{\mathrm{L}} t}$, where only the phase is fluctuating. We shall see below that the fluctuations of the laser phase are "diffusive" – the phase makes a "random walk". In terms of a distribution function $P(\phi, t)$, this behaviour is

described by a diffusion equation,

$$\frac{\partial}{\partial t}P = D\frac{\partial^2}{\partial \phi^2}P \tag{3.37}$$

where D is called "phase diffusion coefficient". You may remember this equation from heat conduction. Its solution, for an initial state with a fixed phase ϕ_0, is given by a gaussian distribution

$$P(\phi, t|\phi_0) = \frac{1}{\sqrt{4\pi D t}}\mathrm{e}^{-(\phi-\phi_0)^2/4Dt}, \qquad t > 0. \tag{3.38}$$

The width of the gaussian is $2Dt$ and increases with time. For very large times, the distribution is completely flat. (This solution neglects the fact that the phase is only defined in the interval $[0, 2\pi]$. See the exercises for this case.) For $t \to 0$, one recovers a δ-function centered at ϕ_0.

With this result, we can compute the temporal correlation function of the laser field,

$$\langle a^\dagger(t+\tau)a(t)\rangle = \bar{n}_{\mathrm{ss}}\,\mathrm{e}^{\mathrm{i}\omega_{\mathrm{L}}\tau}\langle \mathrm{e}^{-\mathrm{i}[\phi(t+\tau)-\phi(t)]}\rangle,$$

which gives us the spectrum by a Fourier transform with respect to τ. We can assume that the initial phase $\phi(t)$ has some value ϕ_0 and that the later phase $\phi = \phi(t+\tau)$ is a random variable with the distribution $P(\phi, \tau|\phi_0)$ of Eq.(3.38). Taking the average, we evaluate one Gaussian integral and get

$$\langle a^\dagger(t+\tau)a(t)\rangle = \bar{n}_{\mathrm{ss}}\mathrm{e}^{\mathrm{i}\omega_L\tau}\int \mathrm{d}\phi\, P(\phi,\tau|\phi_0)\mathrm{e}^{-\mathrm{i}(\phi-\phi_0)} = \bar{n}_{\mathrm{ss}}\mathrm{e}^{\mathrm{i}\omega_L\tau}\mathrm{e}^{-D\tau}.$$

The temporal correlation function thus decays exponentially with a coherence time $\tau_c = 1/D$. This argument applies only for $\tau > 0$. But if the laser state is stationary, we can argue for $\tau < 0$ that

$$\langle a^\dagger(t+\tau)a(t)\rangle = \langle a^\dagger(t)a(t-\tau)\rangle = \langle a^\dagger(t-\tau)a(t)\rangle^* \tag{3.39}$$

by taking the hermitean conjugate of the operator product. The spectrum (3.36) is thus obtained from a "half-sided Fourier transform" which is calculated as

$$S(\omega) = 2\,\mathrm{Re}\int_0^\infty \mathrm{d}\tau\,\mathrm{e}^{-\mathrm{i}\omega\tau}\langle E^*(t+\tau)E(t)\rangle = \frac{2I_{\mathrm{ss}}D}{(\omega-\omega_L)^2 + D^2} \tag{3.40}$$

The laser spectrum is centered at ω_L with a "Lorentzian" lineshape and a width of the order of D. *The laser linewidth is thus limited by the phase diffusion coefficient.*

Fokker-Planck equation

We now have to find an expression for the phase diffusion coefficient. To this end, we shall derive an equation similar to the diffusion equation (3.37). Since this equation deals with a distribution function for the phase, it seems natural to introduce a (quasi-)probability distribution for the laser amplitude. In the quantum optics books, one learns that the P-function (or Sudarshan function) does this job: $P(\alpha, \alpha^*)$ gives the (quasi-) probability that a coherent state $|\alpha\rangle$ occurs in an expansion of the density operator in the basis of coherent states:[3]

$$\rho = \int \mathrm{d}^2\alpha \, P(\alpha, \alpha^*) |\alpha\rangle\langle\alpha|. \tag{3.41}$$

What is the equation of motion for this distribution? From the quantum optics books or as an exercise, you can show that the following replacement table holds when the photon operators act on the projector $|\alpha\rangle\langle\alpha|$

$$\begin{aligned} a\rho &\mapsto \alpha P \\ a^\dagger\rho &\mapsto \left(\alpha^* - \frac{\partial}{\partial\alpha}\right) P \\ \rho a &\mapsto \left(\alpha - \frac{\partial}{\partial\alpha^*}\right) P \\ \rho a^\dagger &\mapsto \alpha^* P. \end{aligned} \tag{3.42}$$

Exercise. Show that these rules are consistent with associative operator products: for example, $(a^\dagger\rho)a$ and $a^\dagger(\rho a)$ are mapped to the same differential operator acting on P.

The equation resulting from (3.31) is

$$\begin{aligned} \frac{\partial}{\partial t} P(\alpha,\, \alpha^*) &= \frac{1}{2}\left\{ \frac{\partial}{\partial\alpha}(\kappa - G)\alpha + \frac{\partial}{\partial\alpha^*}(\kappa - G)\alpha^* \right\} P \\ &\quad + \frac{G}{4}\frac{\partial^2}{\partial\alpha\,\partial\alpha^*} P, \end{aligned} \tag{3.43}$$

where we have yet neglected gain saturation. (The derivatives act on everything to their right, including P.) An approximate way to take it into account is to replace $G \mapsto G(|\alpha|^2) = G_0/(1 + B|\alpha|^2)$. This is actually an approximation because when transforming to the P-representation, one has to neglect some second order and higher order derivatives.

Let us note that the second-order derivative $\partial_\alpha\partial_{\alpha^*}$ in Eq.(3.43) is directly related to the fact that in the quantum description, the operators a and $a^\dagger$ do not commute.

[3]The quasi-probability $P(\alpha)$ is *not* identical to the diagonal matrix element $\langle\alpha|\rho|\alpha\rangle$ which is called the Q-function (or Husimi function). The difference is that for some states, $P(\alpha)$ is not positive everywhere and may not even exist as an ordinary function.

(Their action on a coherent state cannot reduce to multiplication with the numbers α and α^*, but must involve some derivative, as seen in the replacement rules (3.42)). We already suspect that the second-order derivative may have to do with phase diffusion. We see here that it is connected to the discrete nature of the photons. Sometimes, people develop the picture that each photon that is spontaneously emitted by the gain medium contributes to the cavity field a kind of "one-photon field" whose phase is arbitrary. The amplitude of the cavity field thus performs a "random walk" in phase space together with a "deterministic" increase related to "stimuled emission" where the additional photons add up "in phase" with the field.

We now have with (3.43) a partial differential equation for a phase space distribution. It features second-order derivatives like the simple diffusion equation (3.37). Since we are interested in phase diffusion, it seems natural to use polar coordinates $\alpha = r\,\mathrm{e}^{\mathrm{i}\phi}$. You are asked to make this transformation in the exercises and to derive the result

$$\frac{\partial}{\partial t}P(r,\,\phi) = \frac{1}{2r}\frac{\partial}{\partial r}r^2(\kappa - G(r^2))P + \frac{G(r^2)}{4r}\left(\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial^2}{\partial\phi^2}\right)P. \tag{3.44}$$

The steady state solution of this equation does not depend on the phase, but only on the modulus r of the laser amplitude. In the weak saturation limit where $G(r^2) \approx G_0(1 - Br^2)$, it is given by

$$P_{\mathrm{ss}}(r) \approx \mathcal{N}\,\exp\left[-\frac{B}{2}\left(r^2 - \frac{G_0 - \kappa}{G_0 B}\right)^2\right].$$

This shows a maximum at $r^2_{\max} = n_{\max} = (G_0 - \kappa)/G_0 B$ where we recover the formula for the semiclassical steady-state intensity (up to a conversion factor between intensity and photon number). From this distribution, one can check that the fluctuations of the laser intensity (and the field's modulus) are small in the limit $1/B \gg 1$ (large photon number on average, meaning high above threshold). We can thus confirm that if there are fluctuations, they occur dominantly in the phase of the field.

With this argument, we can go back to the diffusion equation (3.44) and identify the phase diffusion coefficient:

$$D = \frac{G(r^2)}{4r^2} \approx \frac{G(n_{\max})}{4n_{\max}}. \tag{3.45}$$

This formula has been derived first in the 1960/70's by Schawlow and Townes. It shows that phase diffusion (and the laser linewidth), well above threshold, decreases inversely proportional to the laser intensity. When the gain is increased, the emission spectrum thus shows an ever growing peak close to the frequency of the cavity mode, that becomes narrower and narrower. This behaviour is often taken as an experimental proof that a laser is operating.

3.8 Master equation and nonlinearity

3.8.1 Motivation

(Material taken from "Laser theory in manifest Lindblad form" (Henkel, 2007). Not covered in SS 2015.)

The quantum theory of a laser is a textbook example of a nonlinear problem that requires techniques from open quantum systems. The key issue is the nonlinearity in the gain of the laser medium, due to saturation, that leads to coupled nonlinear equations already at the semiclassical level. The quantum theory makes things worse by its use of non-commuting operators.

Recall that in the so-called semiclassical theory (see Sec.**??**), the following equation of motion for the intensity I the laser mode can be derived (Sargent III & Scully, 1972; Orszag, 2000):

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -\kappa I + \frac{GI}{1+\beta I} \tag{3.46}$$

where κ is the loss rate, G is the linear gain, and β describes gain saturation for the laser medium. A quantum upgrade of this theory replaces the intensity by the photon number $a^\dagger a$ where the annihilation operator a describes the field amplitude of the laser mode. Mode loss is easy to handle by coupling the laser mode linearly to a mode continuum 'outside' the laser cavity (Walls & Milburn, 1994). This leads to a master equation for the density matrix in so-called Lindblad form [see Eq.(1.174)]

$$\left.\frac{\mathrm{d}\rho}{\mathrm{d}t}\right|_{\mathrm{loss}} = (L\rho L^\dagger - \tfrac{1}{2}\{L^\dagger L,\, \rho\}) \tag{3.47}$$

with a Lindblad operator $L = L_{\mathrm{loss}} = \sqrt{\kappa}\, a$. Linear gain can be handled in the same way, taking $L_{\mathrm{gain},0} = \sqrt{G}\, a^\dagger$, but gain saturation is more tricky. A heuristic conjecture is a Lindblad operator $L_{\mathrm{gain}} = \sqrt{G}\, a^\dagger (1+\beta a^\dagger a)^{-1/2}$. The operator ordering can only be ascertained *a posteriori*, and it is difficult to choose among the replacements $I \mapsto a^\dagger a$, $aa^\dagger$, or $\frac{1}{2}\{a^\dagger a + aa^\dagger\}$. We illustrate this difficulty in Sec.3.6 where the conventional quantum theory of the laser (due to Scully & Lamb) is presented.

In this section, we start with a microscopic model for the pumping process. This is motivated by experiments with so-called micromasers where a (microwave) cavity is crossed by a beam of excited two-level atoms. Nonlinear gain emerges from a treatment beyond second order in the atom-field coupling.

Orszag (2000); Stenholm (1973) analyze a pumping model based on a dilute stream of excited two-level atoms that cross the laser cavity one by one and interact with the laser mode during some randomly distributed interaction time. This model can be largely handled exactly (Briegel & Englert, 1993), even in the presence of incoherent effects like cavity damping, imperfect atom preparation, and frequency-shifting collisions. The setup has become known as the 'micromaser' because of its experimental realization with a high-quality cavity (Meschede & al., 1985; Brune & al., 1987; Raizen & al., 1989). One line of research has focused on the so-called 'strong coupling regime' that permits the laser mode to be driven into non-classical states (Weidinger & al., 1999; Varcoe & al., 2000).

We focus here on the 'weak coupling' regime. On the level of the master equation for the laser mode, this regime corresponds to a small product of coupling constant and elementary interaction time τ so that one can expand in this parameter. Mandel & Wolf (1995); Orszag (2000) consider a coupling to fourth order. For the description of a realistic experiment, one has to average the master equation with respect to a distribution of the interaction time τ (Sec. 3.8). It turns out, however, that the resulting master equation is not of the well-known Lindblad form, although it preserves the trace of the density matrix. This leads to conflicts with the positivity of the density operator, as is known since the original derivation of the master equation by Lindblad and by Gorini et al. (Lindblad, 1976; Gorini & al., 1976). We have shown that this problem can be cured by adding certain terms in sixth order to the master equation (Henkel, 2007). The material presented here is based on this reference. This problem provides us with an example where the Lindblad master equation can be derived from the Kraus-Stinespring representation of the finite-time evolution of the density matrix (Sec. **??**). The mathematical treatment is at the border of validity of the formal Lindblad theory since one has to deal with an infinite-dimensional Hilbert space and continuous sets of Kraus and Lindblad operators.

3.8.2 The micromaser model

Consider a two-level atom with states $|g\rangle$, $|e\rangle$ that is prepared at time t in its excited state $|e\rangle = (1,0)^T$ (density matrix $\rho_A = |e\rangle\langle e|$) and that interacts with a single mode (density matrix ρ) during a time τ. One adopts a Jaynes-Cummings-

Paul Hamiltonian for the atom-field coupling

$$H_{\mathrm{JCP}} = \hbar g \left(a^\dagger \sigma + a \sigma^\dagger \right), \qquad \sigma = |g\rangle\langle e| = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{3.48}$$

(this applies at resonance in a suitable interaction picture). Assume that the initial density operator of the atom+field-system factorizes into $P(t) = \rho(t) \otimes \rho_A$, compute $P(t+\tau)$ by solving the Schrödinger equation and get the following reduced field density matrix (Orszag, 2000; Stenholm, 1973)

$$\rho(t+\tau) = \cos(g\tau\hat{\varphi})\rho(t)\cos(g\tau\hat{\varphi}) + (g\tau)^2 a^\dagger \operatorname{sinc}(g\tau\hat{\varphi})\rho(t)\operatorname{sinc}(g\tau\hat{\varphi})a \tag{3.49}$$

where $\operatorname{sinc}(x) \equiv \sin(x)/x$, and $\hat{\varphi}^2 = aa^\dagger$ is one plus the photon number operator. The operator-valued functions $\cos$ and sinc are defined by their series expansion. Only even powers of the argument occur, hence we actually never face the square root $\hat{\varphi}$ of the operator $aa^\dagger$. In the following, we abbreviate the mapping defined by Eq.(3.49) by $\mathbb{M}_\tau \rho(t)$ (this is sometimes called a superoperator).

The operation (3.49) describes an elementary 'pumping event' of the laser. We assume in the following that this event provides only a small change in the density operator, $(\mathbb{M}_\tau - \mathbb{1})\rho$ is 'small'. To provide a more realistic description, one introduces the following additional averages: excited atoms appear in the laser cavity at a rate r such that $r\tau \ll 1$. Over a time interval Δt, a number of $r\Delta t$ pumping events happens, and the accumulated change in the density operator is $\Delta\rho = r\Delta t(\mathbb{M}_\tau - \mathbb{1})\rho$. This will lead us to a differential equation when Δt is 'small enough'.

We make the additional assumption that the interaction time τ is distributed according to the probability measure $\mathrm{d}p(\tau)$ with mean value $\bar{\tau}$. This reflects the fact that atoms can cross the cavity mode with different velocities, at different positions etc. Also in a conventional laser, atoms interact with the laser mode only during some finite (and randomly distributed) time of the order of the life-time of the excited state. Keeping the coarse-grained time step ($\Delta t \gg \bar{\tau}$), we thus get the difference equation (Orszag, 2000; Stenholm, 1973)

$$\frac{\Delta\rho}{\Delta t} = r \int \mathrm{d}p(\tau)\ (\mathbb{M}_\tau - \mathbb{1})\,\rho. \tag{3.50}$$

To simplify the superoperator appearing on the right hand side, Orszag (2000); Stenholm (1973) suggest an expansion in powers of $g\tau\hat{\varphi}$ up to the fourth order. Using an exponential distribution for $\mathrm{d}p(\tau)$, this leads to the approximate

master equation

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} &= G\left(a^\dagger \rho a - \tfrac{1}{2}\{aa^\dagger,\, \rho\}\right) \\ &\quad + \mathcal{B}\left(3aa^\dagger \rho aa^\dagger + \tfrac{1}{2}\{(aa^\dagger)^2,\, \rho\} - 2a^\dagger\{aa^\dagger,\, \rho\}a\right) \end{aligned} \tag{3.51}$$

where we followed the common practice of interpreting this as a differential equation. We use $\{\cdot,\cdot\}$ to denote the anticommutator. The linear gain is $G = 2r(g\bar{\tau})^2$, and $\mathcal{B} = (g\bar{\tau})^2 G$ is a measure of gain saturation. Indeed, the first line of Eq.(3.51) is in Lindblad form with $L_{\mathrm{gain}} = \sqrt{G}\, a^\dagger$ – applying this operator *increases* the photon number by one. It is easy to see that this leads, on average, to an increasing field amplitude, $(\mathrm{d}/\mathrm{d}t)\langle a\rangle_{\mathrm{gain}} = G\langle a\rangle$.

Losses from the laser mode can be included in the usual way by adding a term of the same structure as the first line of Eq.(3.51), but featuring the Lindblad operator $L_{\mathrm{loss}} = \sqrt{\kappa}\, a$ with the cavity decay rate κ, see Eq.(3.47) (Orszag, 2000; Stenholm, 1973). The same master equation as Eq.(3.51) is also found, using a different pumping model (Mandel & Wolf, 1995).

It is easy to check that Eq.(3.51) preserves the trace of ρ, using cyclic permutations. Nevertheless, it is not of the general form derived by Lindblad for master equations that preserve the complete positivity of density matrices (Lindblad, 1976; Gorini & al., 1976; Alicki & Lendi, 1987). One can show indeed that Eq.(3.51) leads to a density matrix with negative diagonal elements (unphysical for probabilities). Of course, one can accept to work with this kind of 'post-Lindblad' master equations (as they appear frequently in the papers of Golubev and co-workers, see e.g. Golubev & Gorbachev (1986)). It is also possible to construct a set of Lindblad operators $\{L_\lambda\}$ such that with a few additional terms to the master equation (3.51), it can be brought into the Lindblad form. For more details, see "Laser theory in manifest Lindblad form" (Henkel, 2007).

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Chapter 4

Photodetection

4.1 Lecture 19 Jan 15: Glauber's model

For more details, see Sec.4.2. A description based on a general master equation can be found in Sec.4.3.4.

4.2 Photodetector theory à la Glauber

We now have to look in more detail at how exactly the quantum state of the electromagnetic field can be measured. The devices that measure light are called "photodetectors". We shall sketch here the theory that R. Glauber developped to describe these detectors. Light detection has some particular properties: for example, it is practically impossible to resolve the oscillations of the electric field at an optical period of about 10^{-15} s — detectors are simply "too slow" for that. They have a "response time" that is typically in the 10^{-9} s range ("fast" photodetector). In practice (and also in the classical theory), one therefore has only access to time-averaged quantities, the average being taken over at least one optical period.

A good "quantum" model for a light detector is a two-level atom initially in its ground state that gets excited when it is illuminated by light. If we can measure the population of its excited state, we have a signal which is related to the light field. This is the starting point of Glauber's theory.

Detector model. The details are the following: an efficient detection is possible when the two-level system is ionized by the light because the ejected electron can

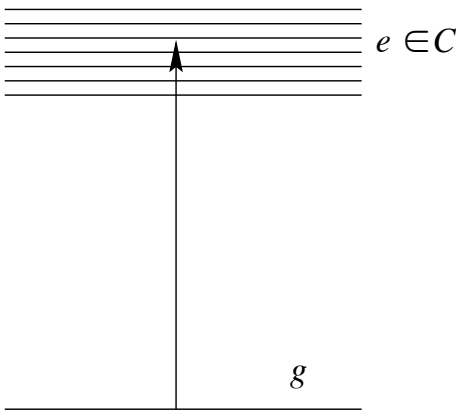

Figure 4.1: Two-level model for a photodetector.

efficiently be detected (this is used in photomultipliers). This means, however, that our excited state is actually a continuum (the electron moves freely), and we cannot use a simple-minded two-state model. So let us assume the level scheme shown in figure 4.3 with a ground state $|g\rangle$ and a continuum C of excited states $|e\rangle$.

We are interested in the probability $p_C(t)$ that this detector goes into any of the excited states $|e\rangle$ ("detection probability"). In terms of the probability amplitudes $c_{e,g}(t)$, the detection probability is given by

$$p_C(t) = \sum_{e \in C} |c_e(t)|^2. \tag{4.1}$$

We start at $t = 0$ with the detector in the ground state. To compute the $c_e(t)$, nothing better can we do than perturbation theory, as we did in chapter 1. The result for the state $|\tilde{\psi}(t)\rangle$ in the interaction picture is

$$|\tilde{\psi}(t)\rangle \approx |\tilde{\psi}(0)\rangle - \frac{\mathrm{i}}{\hbar} \int_0^t \mathrm{d}t'\, V(t') |\tilde{\psi}(0)\rangle \tag{4.2}$$

Projecting this on the final state with one electron excited, we get

$$c_e(t) = -\frac{\mathrm{i}}{\hbar} \int_0^t \mathrm{d}t'\, \langle e; f | V(t') | g; i \rangle\, \mathrm{e}^{\mathrm{i}\omega_{eg} t'} \tag{4.3}$$

where $|i\rangle$ and $|f\rangle$ are the initial and final field states. The interaction potential is taken in the electric dipole coupling

$$V(t) = -\hat{\mathbf{d}} \cdot \mathbf{E}(t) = -\hat{\mathbf{d}} \cdot \mathbf{E}^+(t) - \hat{\mathbf{d}} \cdot \mathbf{E}^-(t) \tag{4.4}$$

where $\mathbf{E}(t)$ is the electric field operator at the location of the detector, and $\mathbf{E}^{\pm}(t)$ its positive and negative frequency part: in the quantized description, the positive frequency part $\mathbf{E}^{+}(t)$ only contains annihilation operators $a_{\mathbf{k}}(t) \propto \mathrm{e}^{-\mathrm{i}\omega_k t}$, and the negative frequency part only creation operators $a^{\dagger}_{\mathbf{k}}(t) \propto \mathrm{e}^{\mathrm{i}\omega_k t}$.

Detection probability. Also in the quantum theory, we can make the resonance approximation (or equivalently, the r.w.a.) and keep in the time-integral (4.3) only those terms that lead to resonant denominators. Since the transition frequency ω_{eg} is positive, this amounts to keeping only the positive frequency part of the electric field operator. We thus have

$$c_e(t) = \frac{\mathrm{i}}{\hbar}\mathbf{d} \cdot \int_0^t \mathrm{d}t' \, \langle f|\mathbf{E}^{+}(t')|i\rangle \, \mathrm{e}^{\mathrm{i}\omega_{eg}t'} \tag{4.5}$$

where we have used the abbreviation $\mathbf{d} = \langle e|\hat{\mathbf{d}}|g\rangle$. For the probability $p_C(t)$, we get an expression involving $\langle i|E_j^{-}(t'')|f\rangle\langle f|E_i^{+}(t')|i\rangle$. At this point, we can perform the sum over the final field states $|f\rangle$ (these are not resolved, so we add the corresponding probabilities). We thus get

$$p_C(t) = \frac{1}{\hbar^2}\sum_{e\in C} d^{j*}d^i \int_0^t \mathrm{d}t' \, \mathrm{d}t'' \, \langle i|E_j^{-}(t'')E_i^{+}(t')|i\rangle \, \mathrm{e}^{\mathrm{i}\omega_{eg}(t'-t'')} \tag{4.6}$$

An explicit calculation of the sum over the continuum E is only possible with a specific model. Let us assume here the following form

$$\sum_{e\in C} d^{j*}d^i \, \mathrm{e}^{\mathrm{i}\omega_e\tau} = D\,\delta^{ij}\delta(\tau). \tag{4.7}$$

This form is plausible when the width $\Delta\omega_e$ of the continuum is sufficiently broad: for τ of the order of the time resolution of the detector, the phase factors $\mathrm{e}^{\mathrm{i}\omega_e\tau}$ then rapidly average out. Note that this behaviour is a consequence of the fact that "photodetectors are slow". The Kronecker δ^{ij} for the field components is plausible when the detector does not discriminate the field polarization. This is a simplifying assumption that we relax later to describe polarization-sensitive detectors.

Putting (4.7) into the detection probability (4.6), we get the basic result

$$p_C(t) = \eta \int_0^t \mathrm{d}t' \, \langle \mathbf{E}^{-}(t') \cdot \mathbf{E}^{+}(t')\rangle, \qquad \eta = \frac{D}{\hbar^2}, \tag{4.8}$$

where we see that a photodetector is sensitive to a "normally ordered" product of electric field operators (annihilation operators to the right and act first). The

average $\langle \cdots \rangle$ is an average over the field state — which can be a quantum average (in a pure quantum state $|i\rangle$) or a combined quantum and classical average. If the field is described by the density operator $\hat{\rho}$, then we obviously have

$$\langle \mathbf{E}^-(t') \cdot \mathbf{E}^+(t') \rangle = \mathrm{tr}\left(\hat{\rho}\, \mathbf{E}^-(t') \cdot \mathbf{E}^+(t') \right).$$

The time-dependence of the operators reminds us that we deal with Heisenberg operators – the density matrix thus specifies the "initial" state (that does not evolve in time).

Photocurrent. The time derivative of the detection probability gives the production rate of photoelectrons, or (up to a factor e) the photocurrent $j(\mathbf{r}, t)$:

$$\begin{array}{rcl} j(\mathbf{r},t) & = & \eta \langle I(\mathbf{r},t) \rangle, \\ I(\mathbf{r},t) & = & :\mathbf{E}^-(\mathbf{r},t) \cdot \mathbf{E}^+(\mathbf{r},t): \end{array} \tag{4.9}$$

The notation $: \cdots :$ means "normal order" (creation operators to the left, annihilation operators to the right and act first). Up to this operator ordering prescription, we have found an expression that resembles the classical time-averaged Poynting vector (recall that it is also proportional to the squared complex field amplitude). Note that the average here is more involved that in the classical theory, since it also has quantum aspects.[1] The proportionality factor η depends on the detector, and theoretical results are often normalized to make it disappear.

If the field is a single plane wave mode, we see that the photocurrent is proportional to the photon number operator:

$$j(t) = \eta_1 \langle a^\dagger(t) a(t) \rangle \tag{4.10}$$

with $\eta_1 = \eta E_{1\mathrm{ph}}^2$. In this context, the factor η_1 is often interpreted as "detection efficiency", i.e., the probability that a photon is detected.

Polarization filtering

Imagine a polarization filter in front of the photodetector. In classical terms, this means that only a component $\mathbf{u}^* \cdot \mathbf{E}$ is transmitted to the detector.[2] In the

[1] A more precise theory is possible at the classical level ("optical coherence theory"): there, the light field is treated as a stochastic field with fluctuations, since it often originates from many uncorrelated sources (classical representation of individual atoms). In the theory, an extra average over these fluctuations then enters. Experimentally, this average enters because detectors are "slow" and cannot resolve these fluctuations.

[2] The complex conjugate $\mathbf{u}^*$ is applicable when a filter for circular polarization is used.

quantum theory, we can describe this by expanding the field $\mathbf{E}(\mathbf{r},t)$ into modes polarized parallel and perpendicular to $\mathbf{u}$. Only the mode with parallel polarization has a mode function that reaches the detector (recall that the calculation of mode functions is a classical problem, independent of the field quantization). So our photocurrent becomes

$$j_{\mathbf{u}}(\mathbf{r},t) = \langle \mathbf{u}\cdot\mathbf{E}^-(\mathbf{r},t)\,\mathbf{u}^*\cdot\mathbf{E}^+(\mathbf{r},t)\rangle$$

Frequency spectrum

What happens with filtering in frequency space? We can in fact use a very similar reasoning: a classical mode at frequency ω will be transmitted to the detector with an efficiency given by the transmission coefficient of the filter. This quantity can be calculated for a cavity, e.g., and shows peaks (maximum transmission) at specific resonance frequencies. Let us focus on one of these frequencies, say ω_c, and suppose that the filter has a very narrow resonance. The transmitted field is then proportional to the time-Fourier transformed electric field at the resonance frequency ω_c:

$$\mathbf{E}^+_{\rm tr}(t) \propto \tilde{\mathbf{E}}^+(\omega_c)\,\mathrm{e}^{-\mathrm{i}\omega_c t}$$

The corresponding photocurrent is then given by

$$\begin{aligned} j_{\omega_c}(t) &\propto \langle \tilde{\mathbf{E}}^-(\omega_c)\cdot\tilde{\mathbf{E}}^+(\omega_c)\rangle \\ &= \int \mathrm{d}t_1\,\mathrm{d}t_2\,\mathrm{e}^{-\mathrm{i}\omega_c(t_1-t_2)}\langle \mathbf{E}^-(t_1)\cdot\mathbf{E}^+(t_2)\rangle \end{aligned} \tag{4.11}$$

If we assume that the field is stationary in time, we can shift the time arguments in the product of field averages and write

$$j_{\omega_c}(t) \propto T\int \mathrm{d}\tau\,\mathrm{e}^{\mathrm{i}\omega_c\tau}\langle \mathbf{E}^-(0)\cdot\mathbf{E}^+(\tau)\rangle \tag{4.12}$$

where T is the measurement time (at least equal to the time needed to build up the field in the cavity). This equation can be generalized to any detector ("spectrometer") that measures a frequency spectrum. Finally, if the field changes slowly (on the timescale of the measurement time T), we can write

$$j_{\omega_c}(t) \propto T\int \mathrm{d}\tau\,\mathrm{e}^{\mathrm{i}\omega_c\tau}\langle \mathbf{E}^-(t-\tau/2)\cdot\mathbf{E}^+(t+\tau/2)\rangle.$$

This quantity is called the 'Wigner transform' of $\langle \mathbf{E}^-(t)\cdot\mathbf{E}^+(t')\rangle$, a function that combines information in real time and in frequency space.

A result similar to Eq.(4.12) appears when we take into account the the excited state continuum in Fig.4.3 is not 'infinitely broad'. A simple generalization of Eq.(4.7) is the following

$$\sum_{e\in C} d^{j*}d^{i}\,\mathrm{e}^{\mathrm{i}\omega_e\tau} = \frac{D\gamma}{2}\delta^{ij}\,\mathrm{e}^{\mathrm{i}\omega_c\tau}\mathrm{e}^{-\gamma|\tau|}. \tag{4.13}$$

where γ is proportional to the spectral width of the continuum states and ω_c their 'center frequency'. When this is put into Eq.(4.6) for the 'click' probability $p_C(t)$, we find

$$p_C(t) = \frac{\eta\gamma}{2}\int_0^t \mathrm{d}t'\,\mathrm{d}t''\;\mathrm{e}^{\mathrm{i}\omega_c(t'-t'')}\mathrm{e}^{-\gamma|t'-t''|}\langle\mathbf{E}^-(t')\cdot\mathbf{E}^+(t'')\rangle, \qquad \eta = \frac{D}{\hbar^2}, \tag{4.14}$$

(See also Eq.(4.55) for an approach based on the master equation for a two-level system.) The time derivative gives the 'rate of clicks' or the photodetector signal. If the measurement time t is much longer than the 'detector response time' $1/\gamma$, then this derivative has a similar form as Eq.(4.12) except for the factor $\mathrm{e}^{-\gamma|t'-t''|}$ that naturally cuts off the integral. (In many cases, the correlation function $\langle\mathbf{E}^-(t')\cdot\mathbf{E}^+(t'')\rangle$ will decay to zero faster.)

4.3 Master equation formulation

(Material not covered in WS 15/16.)

4.3.1 Idea

Many systems in quantum optics are "open" and follow equations of motion that go beyond the Schrödinger equation. This applies to systems for which the Hamiltonian is not completely known or systems in contact with an "environment". The equations of motion can be formulated as so-called master equations (similar to the Schrödinger picture, dynamics for the system density operator) or as so-called Langevin equations (similar to the Heisenberg picture, dynamics of the system operators). These equations are the quantum analogue of kinetic theories that describe the dynamics of a system including different dissipative effects. They provide, among others, the approach of the system towards thermal equilibrium, but also non-equilibrium states that are driven by external forces, the typical example being a laser field.

In quantum optics, the "system" can be an atom, a collection of atoms, or a field mode in a cavity. The system is "open" because it interacts with the "environment" (rest of the world), that is modelled as a continuum of quantized field modes. An atom

decays irreversibly by emitting a photon into previously empty vacuum modes. This phenomenon of "spontaneous emission" does not allow for a description in terms of a Hamiltonian. In fact, the state of the atom does not remain pure, and entropy increases because the photon can be emitted into any direction of space. A cavity mode decays because photons escape through the cavity mirrors, effectively becoming excitations of the "modes outside the cavity". Alternatively, the field energy can be absorbed by the material making up the mirrors.

We focus first on the derivation of a master equation for a two-level atom. We shall find from the general theory two results: (1) the rate of spontaneous emission of an electronically excited state and (2) Glauber's formula for the signal of a photodetectors in terms of normally ordered correlation functions of the electric field operator.

Time scales

The derivation builds on the following hierarchy of time scales:

$$\frac{2\pi}{\omega_{\mathrm{A}}} \leq \tau_{\mathrm{c}} \ll \frac{2\pi}{\Omega} \sim \frac{1}{\gamma} \tag{4.15}$$

where the shortest time scale is the optical period of the light field that is near-resonant with the atomic transition frequency ω_{A}. This is typically smaller than a few fs (femtoseconds) (in the visible range).

The "correlation time" τ_{c} of the electromagnetic field describes qualitatively the time interval over which the quantum fluctuations of the electromagnetic field (that couple significantly to the atom) are "smooth". Two electric field measurements are not significantly "similar" if they are taken at intervals larger than τ_{c}. We give an estimate below, it turns out to be a few times larger than the optical period.

The third time scale is the Rabi period that scales with the inverse of the Rabi frequency of a laser. This depends of course on the laser power (and the atomic transition dipole), but typical values are in the 1–10 ns (nanoseconds) range. Note that this is much longer than the optical period and the field correlation time.

The last time scale is the lifetime of the excited state of the atom, inversely proportional to the spontaneous decay rate γ. This is typically in the 1–10 ns range as well, depending on the atom.

We thus have a situation similar to "Brownian motion": the atom is "shaken around" by the vacuum field fluctuations that vary much faster than the typical dynamics for the atomic state (Rabi oscillations, spontaneous decay). The master equation takes advantage of this separation of time scales to find an equation of motion that can be applied on "slow time scales". As a result, one gets a single equation that describes both, the "Hamiltonian" effects of a laser (Rabi oscillations) and the "dissipative / friction" effects due to vacuum fluctuations.

Hamiltonian

Two-level atom, field, and electric dipole interaction:

$$H = \frac{\hbar\omega_A}{2}\sigma_3 + \sum_k \hbar\omega_k a_k^\dagger a_k + \sum_k \hbar\left(g_k^* a_k^\dagger \sigma + g_k \sigma^\dagger a_k\right) \tag{4.16}$$

neglect the zero-point energy of the field. Coupling constant $\hbar g_k = -\mathbf{d}_{\mathrm{eg}} \cdot \mathbf{f}_k(\mathrm{x}_A)(\hbar\omega_k/2\varepsilon_0)^{1/2}$ with the mode function $\mathbf{f}_k(\mathrm{x})$ evaluated at the position of the atom. Sometimes called 'vacuum Rabi frequency'.

In the following, we also use the (re-scaled) electric field operator

$$E(t) = \sum_k g_k a_k(t) + \mathrm{h.c.} \tag{4.17}$$

Field correlation time

To get an estimate of the field correlation time, we compute the two-time correlation (or coherence) function

$$C(\tau) = \langle E(t+\tau)E(t)\rangle \tag{4.18}$$

If we use the coupling constants g_k to write the mode expansion of the quantized field (this is equivalent to re-scaling the electric field so that it has units of frequency), we get

$$\begin{aligned}\langle E(t+\tau)E(t)\rangle &= \sum_k |g_k|^2 \langle a_k(t+\tau)a_k^\dagger(t)\rangle \\ &= \sum_k |g_k|^2 \,\mathrm{e}^{-\mathrm{i}\omega_k\tau}\end{aligned} \tag{4.19}$$

An explicit calculation using the mode amplitudes in free space leads to an integral of the form

$$C(\tau) \sim \int_0^\infty \mathrm{d}\omega\, \omega^3 \,\mathrm{e}^{-\mathrm{i}\omega_k\tau} \tag{4.20}$$

which looks like a third derivative of a δ-function in τ. Well, not exactly since the integrations starts at $\omega = 0$. Generalizing to finite temperature, the Fourier transform of the correlation function (4.20) becomes

$$C(\omega) \sim \omega^3 \bar{n}(\omega) = \frac{\omega^3}{\exp(\hbar\omega/k_{\mathrm{B}}T) - 1} \tag{4.21}$$

The sign convention of the Fourier transform is chosen here such that $\omega > 0$ corresponds to photon numbers $a^\dagger a$ ('normal order'), while at $\omega < 0$, the anti-normal order $aa^\dagger$ is picked. See Fig.4.2: the spectrum is proportional to $-\omega^3$ for negative frequencies, $-\omega \gg k_{\mathrm{B}}T/\hbar$. Near zero frequency, it has a quadratic behaviour, at positive frequencies,

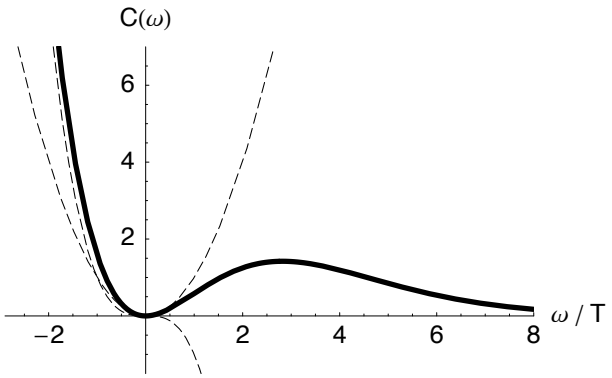

Figure 4.2: Spectrum $C(\omega)$, Eq.(4.21), of the radiation field (normal order) at finite temperature. The frequency is plotted in units of $k_{\rm B}T/\hbar$. The dashed lines are ω^2 and $-\omega^3$.

a maximum near $\omega = 2k_{\rm B}T/\hbar$ (Wien displacement law) and an exponential decay for $\omega \gg k_{\rm B}T/\hbar$.

Actually, we forgot one thing in this estimate: the Hamiltonian we start with is only valid for field modes k near-resonant with the atomic transition, $\omega_k \sim \omega_{\rm A}$. It thus seems reasonable to restrict the frequency integral in Eq.(4.20) to an interval around $\omega_{\rm A}$, with a width $\Delta\omega$ that is typically smaller than $\omega_{\rm A}$.

Now, from the properties of the Fourier transformation, we know that the correlation function $C(\tau)$ given by the integral (4.20) has a "width in time" given by $\Delta\tau\,\Delta\omega \sim 1$. Since this width is precisely the definition of the correlation time, we have

$$\frac{1}{\omega_{\rm A}} \leq \frac{1}{\Delta\omega} \sim \tau_{\rm c}. \tag{4.22}$$

Typical value: a few 10 fs.

4.3.2 Evolution of the atomic operators

We shall work in the Heisenberg picture. Note that operators describing different degrees of freedom (field, atom) commute at equal times.

Atomic dipole

"eliminate" the field operators by solving their equation of motion:

$$\frac{{\rm d}a_k}{{\rm d}t} = \frac{\rm i}{\hbar}\,[H,\, a_k] = -{\rm i}\omega_k a_k - {\rm i}g_k^*\sigma \tag{4.23}$$

$$a_k(t) \;=\; a_k(0)\,\mathrm{e}^{-\mathrm{i}\omega_k t} - \mathrm{i}g_k^* \int_0^t \mathrm{d}t'\,\sigma(t')\,\mathrm{e}^{-\mathrm{i}\omega_k(t-t')} \tag{4.24}$$

the "particular solution" contains the "past" of the atomic dipole operator $\sigma(t')$. Insert this into the equation for σ. Let's look at this equation first:

$$\begin{aligned}\frac{\mathrm{d}\sigma}{\mathrm{d}t} &= \frac{\mathrm{i}}{\hbar}\,[H,\,\sigma] = -\mathrm{i}\omega_\mathrm{A}\sigma + \mathrm{i}\sum_k g_k\left[\sigma^\dagger,\,\sigma\right]a_k \\ &= -\mathrm{i}\omega_\mathrm{A}\sigma + \mathrm{i}\sum_k g_k\sigma_3 a_k \end{aligned}\tag{4.25}$$

We see here that the two-level atom leads to nonlinear equations of motion: the operator product $\sigma_3 a_k$ appears. For the moment, these operators commute (at equal times). But we now want to insert the solution (4.24) for $a_k(t)$, and the two terms that appear here do not commute separately with σ_3. For this reason, we take now a specific operator order (so-called "normal order") where the annihilation operators (a_k or σ) act first. This is the order already used in Eq.(4.25), and we thus get

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = -\mathrm{i}\omega_\mathrm{A}\sigma + \mathrm{i}\sigma_3\mathcal{E}_0(t) + \int_0^t \mathrm{d}t' \sum_k |g_k|^2 \mathrm{e}^{-\mathrm{i}\omega_k(t-t')}\sigma_3(t)\sigma(t') \tag{4.26}$$

where we used the abbreviation

$$\mathcal{E}_0(t) = \sum_k g_k a_k(0)\,\mathrm{e}^{-\mathrm{i}\omega_k t} \tag{4.27}$$

for the freely evolving electric field operator (positive frequency component only, and rescaled by the atomic transition dipole). In the integral over t', we see that the correlation function $C(t-t')$ appears, and using $\tau = t - t'$ as integration variable, we have

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = -\mathrm{i}\omega_\mathrm{A}\sigma + \mathrm{i}\sigma_3\mathcal{E}_0(t) + \int_0^t \mathrm{d}\tau\, C(\tau)\sigma_3(t)\sigma(t-\tau) \tag{4.28}$$

Now comes the *key observation*: under the time integral occur two very different functions. The correlation function $C(\tau)$ is very narrow in τ. The atomic operator $\sigma(t-\tau)$ contains a "fast free evolution" (generated by the first term in its equation of motion), but once this is factored out, we anticipate that its evolution is "slow":

$$\sigma(t-\tau) = \mathrm{e}^{-\mathrm{i}\omega_\mathrm{A}(t-\tau)}\tilde{\sigma}(t-\tau) \approx \mathrm{e}^{-\mathrm{i}\omega_\mathrm{A}(t-\tau)}\tilde{\sigma}(t) = \mathrm{e}^{+\mathrm{i}\omega_\mathrm{A}\tau}\sigma(t) \tag{4.29}$$

Here, we introduced temporarily the dipole operator $\tilde{\sigma}(t')$ (in an interaction picture).

The main benefit of this approximation is that only atomic operators at time t appear in the equation of motion. The two-level commutation rules[3] give $\sigma_3(t)\sigma(t) = -\sigma(t)$, and we end up with

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = -\mathrm{i}\omega_\mathrm{A}\sigma + \mathrm{i}\sigma_3\mathcal{E}_0(t) - \sigma(t)\int_0^t \mathrm{d}\tau\, C(\tau)\,\mathrm{e}^{\mathrm{i}\omega_\mathrm{A}\tau} \tag{4.30}$$

The upper integration limit t is actually the difference between our initial time and the actual time where the equation of motion is computed. We now let this time difference be much larger than the correlation time τ_c. This is consistent with the assumption that the atomic dynamics is slow on the scale of the field's correlation time. Then the integrand is effectively zero at the upper limit, and we get a constant complex number

$$\gamma + \mathrm{i}\delta\omega_\mathrm{A} = \int_0^{t\gg\tau_\mathrm{c}} \mathrm{d}\tau\, C(\tau)\,\mathrm{e}^{\mathrm{i}\omega_\mathrm{A}\tau} = \frac{S(\omega_\mathrm{A})}{2} + \mathrm{i}\mathcal{P}\int_{-\infty}^{+\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{S(\omega)}{\omega-\omega_\mathrm{A}} \tag{4.31}$$

where $S(\omega)$ is the Fourier transform of the correlation function $C(\tau)$ and $\mathcal{P}$ means the principal part of the integral.

Spontaneous decay rate and Lamb shift

Explicit calculation: spectrum of vacuum fluctuations

$$S(\omega) = 2\pi\sum_k |g_k|^2\delta(\omega_k-\omega) \tag{4.32}$$

Take a plane wave expansion and sum over the polarization vectors $\mathbf{u}_{\mathbf{k}\lambda}$ in the coupling constants g_k

$$\sum_\lambda |\mathbf{d}_\mathrm{eg}\cdot\mathbf{u}_{\mathbf{k}\lambda}|^2 = |\mathbf{d}_\mathrm{eg}|^2 - |\mathbf{d}_\mathrm{eg}\cdot\hat{\mathbf{k}}|^2 \tag{4.33}$$

where $\hat{\mathbf{k}}$ is the unit vector along $\mathbf{k}$. This formula arises because the $\mathbf{u}_{\mathbf{k}\lambda}$ are perpendicular to $\mathbf{k}$. Integration over the angles of $\mathbf{k}$ gives

$$\int \mathrm{d}\Omega(\hat{\mathbf{k}})\left(|\mathbf{d}_\mathrm{eg}|^2 - |\mathbf{d}_\mathrm{eg}\cdot\hat{\mathbf{k}}|^2\right) = 4\pi|\mathbf{d}_\mathrm{eg}|^2 - \frac{4\pi}{3}|\mathbf{d}_\mathrm{eg}|^2 = \frac{8\pi}{3}|\mathbf{d}_\mathrm{eg}|^2 \tag{4.34}$$

The integral over the length of $\mathbf{k}$ is trivial because of the δ-function in Eq.(4.32), its length is fixed to $|\mathbf{k}| = \omega/c$. Putting everything together, we get

$$S(\omega) = \frac{2\pi}{3\hbar^2}|\mathbf{d}_\mathrm{ge}|^2\frac{\hbar\omega}{2\varepsilon_0}\frac{8\pi\omega^2}{(2\pi c)^3} \tag{4.35}$$

[3] A pedantic remark: operator products evolve as products under the Heisenberg equations of motion. This is because taking the commutator with a product is compatible with the product rule of (time) derivatives.

where the last factor is the density of field modes per $\mathrm{d}\omega$ and volume. We can thus say that the spontaneously decaying atom is a "detector" for vacuum field fluctuations. The decay rate γ

$$\gamma = \frac{1}{2} S(\omega_{\mathrm{A}}) = \frac{|\mathbf{d}_{\mathrm{ge}}|^2 (\omega_{\mathrm{A}}/c)^3}{6\pi\hbar\varepsilon_0} \tag{4.36}$$

is also called the "natural linewidth" of the atomic transition $|\mathrm{g}\rangle \leftrightarrow |\mathrm{e}\rangle$ because it gives the width in frequency of the spontaneous emission spectrum. In order of magnitude, with $|\mathbf{d}_{\mathrm{ge}}| \sim e a_0$ (electron charge $\times$ atom size)

$$\frac{\gamma}{\omega_{\mathrm{A}}} \sim \alpha_{\mathrm{fs}} (a_0/\lambda_{\mathrm{A}})^2 \sim \alpha_{\mathrm{fs}}^3 \tag{4.37}$$

with fine structure constant $\alpha_{\mathrm{fs}} = e^2/4\pi\varepsilon_0\hbar c \approx 1/137.04$ and wavelength λ_{A} of resonant transition. Hence, indeed decay is "slow" on the scale of the optical period.

Frequency shift $\delta\omega_{\mathrm{A}}$ is related to asymmetry of vacuum spectrum around the transition frequency. Interpretation from second order perturbation theory: modes slightly below the atomic resonance, $\omega_k \leq \omega_{\mathrm{A}}$ tend to push the level $|\mathrm{e}\rangle$ upwards. Modes above resonance: push downwards. Null effect if spectrum is flat.

Actual calculation requires UV cutoff ω_{uv} and interaction Hamiltonian beyond the resonance (rotating wave) approximation (Hans Bethe $\sim$ 1948, calculation of the Lamb shift). Order of magnitude:

$$\delta\omega_{\mathrm{A}} \approx \gamma \log(\omega_{\mathrm{uv}}/\omega_{\mathrm{A}}) \tag{4.38}$$

and a consistent cutoff is the electron rest mass, $\omega_{\mathrm{uv}} \sim m_{\mathrm{e}} c^2/\hbar$. Hence comparable to the natural linewidth. Note: relativistic theory required, where wavelengths up to electron Compton wavelength $\hbar/m_{\mathrm{e}}c$ are involved. This goes beyond the "long wavelength approximation" because $\hbar/m_{\mathrm{e}}c \sim \alpha_{\mathrm{fs}} a_0 \ll a_0$.

Atomic populations

Actually, we did not yet show that γ is the rate of decay for the excited state. For this, we need the equation of motion for the occupations of the two energy levels. This is described by the atomic operator σ_3, also called the "inversion" because $\langle\sigma_3\rangle > 0$ when the excited state is more occupied than the ground state.

Heisenberg equation of motion

$$\begin{aligned} \frac{\mathrm{d}\sigma_3}{\mathrm{d}t} &= +2\mathrm{i} \sum_k \left[g_k^* a_k^\dagger \sigma - g_k \sigma^\dagger a_k \right] \\ &= 2\mathrm{i} \left[\mathcal{E}_0^\dagger(t)\sigma - \sigma^\dagger \mathcal{E}_0(t) \right] \\ &\quad - 2 \int_0^t \mathrm{d}t' \left[C^*(t-t')\sigma^\dagger(t')\sigma(t) + C(t-t')\sigma^\dagger(t')\sigma(t) \right] \end{aligned} \tag{4.39}$$

where we inserted the formal solution for $a_k(t)$ and brought the operator products in normal order. We apply to the t'-integral the same prescription as before and get

$$\int_0^t \mathrm{d}t' \left[C^*(t-t')\sigma^\dagger(t')\sigma(t) + C(t-t')\sigma^\dagger(t')\sigma(t)\right]$$
$$\approx (\gamma - \mathrm{i}\delta\omega_\mathrm{A})\sigma^\dagger(t)\sigma(t) + (\gamma + \mathrm{i}\delta\omega_\mathrm{A})\sigma^\dagger(t)\sigma(t) = \gamma(\sigma_3(t) + \mathbb{1}) \qquad (4.40)$$

where the frequency shift drops out. By construction, the operator $\sigma_3(t) + \mathbb{1}$ gives the occupation of the excited state. From Eqs.(4.39, 4.40), we can thus read off the decay rate 2γ for the excited state population, while the ground state remains stable.

Finally, the equation for the inversion operator becomes

$$\frac{\mathrm{d}\sigma_3}{\mathrm{d}t} = -2\gamma(\sigma_3(t) + \mathbb{1}) + 2\mathrm{i}\left[\mathcal{E}_0^\dagger(t)\sigma - \sigma^\dagger\mathcal{E}_0(t)\right] \qquad (4.41)$$

Combined with the equation of motion for the atomic dipole operator,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}t} = -(\gamma + \mathrm{i}\omega_\mathrm{A})\sigma + \mathrm{i}\sigma_3\mathcal{E}_0(t) \qquad (4.42)$$

we have thus found the "optical Bloch equations".

In Eq.(4.42), we have made the replacement $\omega_\mathrm{A} + \delta\omega_\mathrm{A} \mapsto \omega_\mathrm{A}$ for the atomic frequency. This is called "renormalization": we combine the shift induced by the coupling to the vacuum field with the "naked" transition frequency into the frequency that can be physically observed. Recall that in reality, we can never "switch off" the coupling to the vacuum. Hence the "naked" two-level atom that we started with is actually a theoretical artefact.

4.3.3 Application 1: Bloch equations

Note that we did not specify yet the state of the light field: it is in fact encoded in the operator $\mathcal{E}_0(t)$ that depends on the initial field operators $a_k(0)$. Two examples will be studied now, the first one being an atom driven by a laser field. We have argued that to a good approximation, we can assume that the light field in a coherent state $|\alpha_\mathrm{L}\rangle$. We assume that at $t = 0$, the total system is in the product state $|\psi(0), \alpha_\mathrm{L}\rangle$ and take the expectation value of the Bloch equation. This gives the dynamics of the Bloch vector as follows (optical Bloch equations in the proper sense)

$$\frac{\mathrm{d}s}{\mathrm{d}t} = -(\gamma + \mathrm{i}\omega_\mathrm{A})s + (\mathrm{i}/2)s_3\Omega\,\mathrm{e}^{-\mathrm{i}\omega_\mathrm{L}t} \qquad (4.43)$$
$$\frac{\mathrm{d}s_3}{\mathrm{d}t} = -2\gamma(s_3(t) + 1) + \mathrm{i}\left[\Omega^*(t)\mathrm{e}^{\mathrm{i}\omega_\mathrm{L}t}s - s^*\Omega\mathrm{e}^{-\mathrm{i}\omega_\mathrm{L}t}\right] \qquad (4.44)$$

where $\Omega/2 = \langle\alpha_\mathrm{L}|\mathcal{E}_0(0)|\alpha_\mathrm{L}\rangle$ is the (complex) Rabi frequency and ω_L the frequency of the laser mode.

These equations have time-dependent coefficients, but this can be removed by making a transformation into a "rotating frame". We make the replacement

$$s(t) \mapsto s(t)\mathrm{e}^{-\mathrm{i}\omega_{\mathrm{L}}t} \tag{4.45}$$

where the "new" $s(t)$ satisfies Bloch equations with time-independent coefficients

$$\boxed{\begin{array}{rcl} \dfrac{\mathrm{d}s}{\mathrm{d}t} & = & -(\gamma - \mathrm{i}\delta)s + (\mathrm{i}/2)s_3\Omega \\ \dfrac{\mathrm{d}s_3}{\mathrm{d}t} & = & -2\gamma(s_3(t)+1) + \mathrm{i}\left[\Omega^* s - s^*\Omega\right] \end{array}} \tag{4.46}$$

where $\delta = \omega_{\mathrm{L}} - \omega_{\mathrm{A}}$ is the "laser detuning". Note that the symbols are not the same throughout the books: the Rabi frequency Ω or the decay rate γ can differ by a factor of 2 (or -2), the detuning can have the opposite sign.

The Bloch equations are a "workhorse" of atomic physics and quantum optics. They are used to compute light absorption, excitation spectra, population transfer, radiation forces on atoms etc. In the exercises, you compute the stationary state of the Bloch equations (attention with the different sign for δ: this one should be correct)

$$s_{\mathrm{ss}} = \frac{-\mathrm{i}(\Omega/2)(\gamma + \mathrm{i}\delta)}{\gamma^2 + \delta^2 + \Omega^2/2} \tag{4.47}$$

$$s_{3,\mathrm{ss}} = \frac{-(\gamma^2 + \delta^2)}{\gamma^2 + \delta^2 + \Omega^2/2} \tag{4.48}$$

Discussion: average atomic dipole operator (induced by laser field), average inversion. Line broadening.

Exercise: total excitation N, does not commute when laser included. Expectation value of $\dot{N}$ in stationary state, interpretation as total scattered intensity.

Exercise: spectrum of spontaneous emission, from formal solution. Need formal solution of atomic dipole operator, Eq.(4.53).

4.3.4 Application 2: the Glauber photodetector

Roy Glauber (Nobel prize 2005) developped in the 1960s the theory of photodetection. His main result is that the signal of a photodetector is proportional to

$$I(t) \propto \langle E^{(-)}(t)E^{(+)}(t)\rangle \tag{4.49}$$

where $E^{(+)}(t)$ is the positive frequency part of the electric field operator at the detector position. This signal is constructed in such a way that if the field is in the vacuum state, a detector gives no signal: perfectly reasonable. But due to the presence of vacuum

fluctuations (also nonzero expectation values of a product of field operators!), not easy to implement in the theory.

We can recover the relevant features of Glauber's theory with our two-level atom. We shall actually show that under suitable approximations,

$$I(t) \propto \langle \mathcal{E}_0^\dagger(t)\mathcal{E}(t)\rangle \tag{4.50}$$

where $\mathcal{E}(t)$ is the re-scaled electric field operator we introduced in Eq.(4.27). Note that it contains positive frequency components only, and also only those modes that are near-resonant with the atomic transition frequency $\omega_{\rm A}$. Indeed, Glauber's model for a photodetector is a two-state system that is prepared in the ground state. Incident light can be absorbed, leading to some population in the excited state. This population is then "rapidly removed" from the system. A physical example: the excited atom is ionized and the free electron moves away (it cannot come back to recombine into the ground state). This is actually the process that happens in a photomultiplier ("avalanche photodiode").

So let us see what our Heisenberg equations (4.41, 4.42) give when the atom in the ground state and the field is in an arbitrary state. We are interested in the rate of change of the excited state population:

$$I := \frac{{\rm d}p_e}{{\rm d}t} = \frac{\rm d}{{\rm d}t}\langle\frac{\sigma_3+\mathbb{1}}{2}\rangle = \frac{1}{2}\frac{{\rm d}\langle\sigma_3\rangle}{{\rm d}t} \tag{4.51}$$

Since the atom is in the ground state, the operator $\sigma_3+\mathbb{1}$ that appears in Eq.(4.41) averages to zero. We are left with

$$I = {\rm i}\langle\mathcal{E}_0^\dagger(t)\sigma(t) - \sigma^\dagger(t)\mathcal{E}_0(t)\rangle \tag{4.52}$$

and insert the formal solution for the atomic dipole operator [similar to Eq.(4.24)]:

$$\sigma(t) = \sigma(0)\,{\rm e}^{-(\gamma+{\rm i}\omega_{\rm A})t} + {\rm i}\int_0^t {\rm d}t'\,\sigma_3(t')\mathcal{E}_0(t')\,{\rm e}^{-(\gamma+{\rm i}\omega_{\rm A})(t-t')} \tag{4.53}$$

This gives

$$\begin{aligned} I &= -\int_0^t {\rm d}t'\left[\langle\mathcal{E}_0^\dagger(t)\sigma_3(t')\mathcal{E}_0(t')\rangle\,{\rm e}^{-(\gamma+{\rm i}\omega_{\rm A})(t-t')} + {\rm h.c.}\right] \\ &= \int_0^t {\rm d}\tau\left[\langle\mathcal{E}_0^\dagger(t)\mathcal{E}_0(t-\tau)\rangle\,{\rm e}^{-(\gamma+{\rm i}\omega_{\rm A})\tau} + {\rm h.c.}\right] \end{aligned} \tag{4.54}$$

where in the second line, we used the approximation that the inversion $\sigma_3(t')$ is evolving slowly and took its expectation value in the initial state (atom in the ground state, not correlated with the field). If we also assume that the expectation value evolves slowly

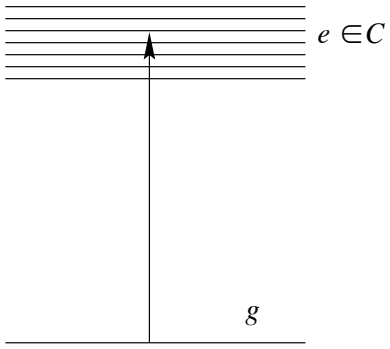

Figure 4.3: Two-level model for a fast photodetector. The signal corresponds to a sum of transition rates into all components of the upper state (quasi)continuum.

with time t, $\langle \mathcal{E}_0^\dagger(t)\mathcal{E}_0(t-\tau)\rangle \approx \langle \mathcal{E}_0^\dagger(t+\tau)\mathcal{E}_0(t)\rangle$ [this is strictly true for a stationary field, but stationary fields do not give not very interesting signals on a photodetector], we can combine the '$+\mathrm{h.c.}$' term into the single integral

$$I = \int_{-t}^{t} \mathrm{d}t' \langle \mathcal{E}_0^\dagger(t)\mathcal{E}_0(t+\tau)\rangle \, \mathrm{e}^{-\gamma|\tau|+\mathrm{i}\omega_\mathrm{A})\tau} \tag{4.55}$$

We observe that the photodetector signal is similar to the Fourier transform of the two-time field correlation function. We can already confirm that the signal is given by a normally ordered expectation value of field operator. Let us consider two limits.

Narrow-band detector

If the decay time $1/\gamma$ is "very long" (compared to the correlation time of the field), the photodetector signal is essentially given by the field spectrum, taken at the transition frequency ω_A. Indeed, the τ-integral in Eq.(4.55) picks from the positive frequency operator $\mathcal{E}_0(t+\tau)$ those components that evolve like $\mathrm{e}^{-\mathrm{i}\omega_\mathrm{A})\tau}$. A narrow band photodetector is thus simply a 'spectrometer', and the quantity it measures is the Fourier transform of the temporal correlation (or coherence) function. Experimentally, this situation can be achieved by placing a narrow-band frequency filter in front of a photodetector.

Fast detector

In some photodetectors, the excited state has a very short lifetime $1/\gamma$. This happens for example when the excited state actually ionizes and gives away its electron. The excitation rate I is then actually an integral over all components of the excited state 'continuum', as sketched in Fig.4.3. Under the integral, the exponential $\mathrm{e}^{-\gamma|\tau|}$ becomes

very short-ranged in τ, and we can make the replacement

$$\mathrm{e}^{-\gamma|\tau|} \mapsto \frac{2}{\gamma}\delta(\tau) \tag{4.56}$$

This leads to Glauber's formula

$$I(t) = \eta\langle\mathcal{E}_0^\dagger(t)\mathcal{E}_0(t)\rangle \tag{4.57}$$

where the expectation value of the instantaneous intensity operator $\mathcal{E}_0^\dagger(t)\mathcal{E}_0(t)$ appears, in normal order, of course. The prefactor η actually is a number characteristic for the detector and is called "quantum efficiency". It must be determined experimentally. If the intensity is scaled to "photons per second", then η gives the detection probability per photon, and I the "rate of detected photons per second".

Intensity (photon) correlations

Generalization to multiple detection events: $I(t)$ interpreted as probability per unit time to detect one photon at time t. Probability to detect one photon at time t_1 and another one at t_2:

$$I(t_1, t_2) = \eta^2\langle\mathcal{E}_0^\dagger(t_1)\mathcal{E}_0^\dagger(t_2)\mathcal{E}_0(t_2)\mathcal{E}_0(t_1)\rangle, \qquad t_2 > t_1 \tag{4.58}$$

time- and normal-ordered operator product. Interpretation.

Temporal coherence

Definition in terms of factorization of $\langle\mathcal{E}_0^\dagger(t)\mathcal{E}_0(t')\rangle$ (first-order temporal coherence). And for second-order.

Examples: monochromatic field always first-order coherent. Black-body radiation not. Two-photon state is second-order coherent, but not in first-order.

Bibliography