Einführung in die Quantenoptik I

Wintersemester 2015/16
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Übungsaufgaben Blatt 2

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Problem 2.1 – Electric dipoles in hydrogen (10 points)

In the lecture, we have seen the matrix elements of the dipole operator and the corresponding selection rules. In this problem, you work out some details in the Hydrogen atom.

(i) Write down the action of the electric dipole operator $\mathbf{d}$ on the wave functions of the Hydrogen atom. [Bonus points: any corrections that appear because of the motion of the proton (keyword 'reduced mass')?]

(ii) Look up the spherical harmonics (*Kugelflächenfunktionen*) and show that the dipole matrix elements vanish between s-states:

$$\langle n\text{s}0|\mathbf{d}|n'\text{s}0\rangle = \mathbf{0} \tag{2.1}$$

where the notation $\langle \mathbf{r}|nlm\rangle = \psi_{nlm}(\mathbf{r})$ has been used. Treat separately the (xy)- and the z-components of $\mathbf{d}$.

(iii) Show that the same is true for the z-component d_z between the states $|n\text{s}0\rangle$ and $|n\text{p}{\pm}1\rangle$.

(iv) A few of the nonzero dipole matrix elements have the values

$$\langle 2\text{p}0|d_z|1\text{s}0\rangle = \frac{128\sqrt{2}}{243} e a_0\,, \qquad \langle 2\text{p}1|(d_x + \mathrm{i} d_y)|1\text{s}0\rangle = -\frac{256}{243} e a_0 \tag{2.2}$$

You are invited to check one of these.

(v) Consider a Hydrogen atom prepared in the superposition state

$$|\psi(t)\rangle = \frac{1}{\sqrt{2}}\left\{ \mathrm{e}^{-\mathrm{i}E_1 t/\hbar}|1\text{s}0\rangle - \mathrm{e}^{-\mathrm{i}E_2 t/\hbar}|2\text{p}0\rangle \right\} \tag{2.3}$$

where E_n $(n = 1, 2)$ are the energy levels. (This state is a solution of the time-dependent Schrödinger equation, right?) Compute the expectation value

$$\langle d_z(t)\rangle := \langle \psi(t)|d_z|\psi(t)\rangle \tag{2.4}$$

and show that it oscillates at the Bohr frequency $\omega_{21} = (E_2 - E_1)/\hbar$. This formula provides a link between quantum mechanics and electrodynamics: argue why it could be a key result in Sommerfeld's classic *Atombau und Spektrallinien*.

Problem 2.2 – Material polarisation (5 points)

Look up numbers for the refractive index of glass (take your spectacles, for example). Write down a formula that gives the polarisation field in glass when an electromagnetic wave (intensity $1\,\mathrm{mW/cm^2}$, say) is traversing it. Guess a typical number density of ‘polarisable atoms’ in glass and make an estimate for the electric dipole moment per atom. Check that this is very small when converted into the displacement of an electron charge. What else could carry an electric dipole moment in glass?

Problem 2.3 – Minimal coupling and long-wavelength approximation (5 points)

For a system of charged particles, the Hamiltonian is given by (look up your electrodynamics lecture)

$$H = \sum_\alpha \frac{(\mathbf{p}_\alpha - e_\alpha \mathbf{A}(\mathbf{r}_\alpha))^2}{2m_\alpha} + e_\alpha \phi(\mathbf{r}_\alpha) \tag{2.5}$$

(i) Focus on a system with a single ‘mobile’ electron where the other charges are described by the (static) charge density ρ. Show that in the electromagnetic field of a laser, we may use in Eq.(2.5) the potentials

$$\phi = \int \mathrm{d}^3 r' \frac{\rho(\mathbf{r}')}{4\pi\varepsilon_0 |\mathbf{r} - \mathbf{r}'|} + \phi_L(\mathbf{r}, t)\,, \qquad \mathbf{A} = \mathbf{A}_L(\mathbf{r}, t) \tag{2.6}$$

(ii) The same formalism is used in quantum mechanics where H becomes an operator acting on the wave function ψ. Imagine that an alternative wave function ψ' is introduced by multiplying with a phase factor (‘unitary transformation’)

$$\psi'(\mathbf{r}, t) = \mathrm{e}^{\mathrm{i}\alpha(\mathbf{r},t)} \psi(\mathbf{r}, t) \tag{2.7}$$

Show that ψ' solves a different time-dependent Schrödinger equation given by

$$\mathrm{i}\hbar \partial_t \psi' = \left(\mathrm{e}^{\mathrm{i}\alpha(\mathbf{r},t)} H \mathrm{e}^{-\mathrm{i}\alpha(\mathbf{r},t)} \right) \psi' - \hbar \partial_t \alpha(\mathbf{r}, t) \psi' \tag{2.8}$$

(iii) Consider an electron bound around the origin whose orbital has a typical size a much smaller than the wavelength λ (scale on which ϕ_L and $\mathbf{A}_L$ vary spatially). Check that with the choice $\alpha(\mathbf{r}, t) = (e/\hbar) \int^t \mathrm{d}t' \phi_L(\mathbf{0}, t') - e\mathbf{r} \cdot \mathbf{A}_L(\mathbf{0}, t)/\hbar$, the Schrödinger equation (2.8) takes the approximate form

$$\mathrm{i}\hbar \partial_t \psi' = \left(\frac{\mathbf{p}^2}{2m} + V_C(\mathbf{r}) \right) \psi' - e\mathbf{r} \cdot \mathbf{E}_L(\mathbf{0}, t) \psi' + \mathcal{O}(a/\lambda) \tag{2.9}$$

where the Coulomb potential V_C arises from the first term in the electric potential (2.6).

No guarantee for correct signs.