Einführung in die Quantenoptik I

Wintersemester 2015/16
Carsten Henkel

Übungsaufgaben Blatt 3 Ausgabe: 05. Nov 2015
Abgabe: 17. Nov 2015

Problem 3.1 – An atom and a weak laser (7 points)

In the lecture, we have discussed the Schrödinger equation for a two-level atom and found for the two amplitudes $\tilde{c}_e$ and $\tilde{c}_g$ the system

$$\begin{aligned} \mathrm{i}\partial_t \tilde{c}_e &= -\frac{\omega_A - \omega}{2}\tilde{c}_e + \Omega\cos(\omega_L t)\,\mathrm{e}^{\mathrm{i}\omega t}\tilde{c}_g \\ \mathrm{i}\partial_t \tilde{c}_g &= +\frac{\omega_A - \omega}{2}\tilde{c}_g + \Omega\cos(\omega_L t)\,\mathrm{e}^{-\mathrm{i}\omega t}\tilde{c}_e \end{aligned} \tag{3.1}$$

At this stage, the resonance approximation has not yet been applied. (i) Recall the meaning of the symbols used here. (ii) You are invited to take the 'other' route than in the lecture and choose the frequency $\omega = \omega_A$ for the 'rotating frame'. Assume that the laser is weak and show that the expansion in powers of Ω has as lowest terms:

$$\begin{aligned} \tilde{c}_a^{(0)}(t) &= \tilde{c}_a^{(0)}(0)\,, \qquad a = e, g \\ \tilde{c}_e^{(1)}(t) &= -\mathrm{i}\Omega\int_0^t \mathrm{d}t'\,\cos(\omega_L t')\,\mathrm{e}^{\mathrm{i}\omega_A t'}\tilde{c}_g^{(0)}(t') \\ &\ldots \end{aligned} \tag{3.2}$$

(iii) Work out the integral in Eq.(3.2) and compare to the result of the lecture ($R = \sqrt{\Omega^2 + \Delta^2}$)

$$\text{lecture:} \qquad \tilde{c}_e(t) = -\mathrm{i}\frac{\Omega}{R}\sin(Rt/2) \tag{3.3}$$

by taking the weak-field approximation of Eq.(3.3).

Attention: the amplitude in Eq.(3.3) is written in the frame where $\omega = \omega_L$, but the translation into another frame is easy, right?

Problem 3.2 – Bloch vectors (7 points)

In the lecture, we have introduced the Bloch vector for a two-level atom from the average of the vector of Pauli matrices:

$$\mathbf{s} = \langle\psi|\boldsymbol{\sigma}|\psi\rangle \tag{3.4}$$

where $|\psi\rangle$ is the state of the atom. (i) Expand $|\psi\rangle$ over the basis states $|e\rangle$ and $|g\rangle$ and give expressions for the three components of s in terms of c_e and c_g. Show for example that

$$s_3 = |c_e|^2 - |c_g|^2 \tag{3.5}$$

(ii) Show that in this situation, the Bloch vector has unit length: $\mathrm{s}^2 = s_1^2 + s_2^2 + s_3^2 = 1$. (iii) Show that this is *not* the same as the expectation value of $\sigma_1^2 + \sigma_2^2 + \sigma_3^2$. (iv) Argue that any Hamiltonian that acts on the Hilbert space of a two-level system is a linear combination of Pauli matrices (well, excluding a 'trivial constant'). (v) Consider two quantum states $|\psi\rangle$ and $\chi\rangle$ with the Bloch vectors s and $\mathbf{v}$. Show that their overlap satisfies the identity

$$|\langle\chi|\psi\rangle|^2 = \frac{\mathbf{v}\cdot\mathbf{s}+1}{2} \tag{3.6}$$

so that orthogonal states must have Bloch vectors with opposite sign, but Bloch vectors that are orthogonal correspond to states with a partial overlap.

Problem 3.3 – Dephasing, lifetime, decay and so on (5 points)

In the context of spin resonance or two-level systems in general, one introduces two different time scales, called T_1 and T_2. One time (longer) is relevant for the lifetime of the excited state, while the second time (shorter) governs the 'coherence' of the system and depends on mechanisms like 'dephasing'. Other people talk of 'free induction decay' for the shorter time. Try to find estimates for numbers in typical systems. Try to find the longest coherence time: is there an upper limit like $T_2 \le T_1$ (or $T_1 \le T_2$)? What do people mean by T_2^* or T_1^*?