Einführung in die Quantenoptik I

Wintersemester 2015/16
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Übungsaufgaben Blatt 4

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Problem 4.1 – Bloch vector, dipole moment, polarization field (10 points)

(1) The rotating frame introduced in the lecture simplifies the equations of motion for a two-level system. For the density operator ρ, it corresponds to a time-dependent unitary transformation

$$\rho(t) = U(t)\tilde{\rho}(t)U^{\dagger}(t)\,, \qquad U(t) = \exp(-\mathrm{i}\sigma_3 \omega t/2) \tag{4.1}$$

Use the formula for the exponential of a Pauli matrix to show that the expectation value of the dipole operator transforms in the rotating frame into

$$\langle \mathbf{d} \rangle = \mathbf{d}_{ge} \langle \tilde{\sigma}_1(t) \cos \omega t - \tilde{\sigma}_2(t) \sin \omega t \rangle \tag{4.2}$$

where $\mathbf{d}_{ge}$ is the (fixed) dipole matrix element and the expectation values $\langle \tilde{\sigma}_{1,2}(t) \rangle$ are computed with the help of $\tilde{\rho}(t)$.

(2) The full set of Bloch equations (in the rotating frame with $\omega = \omega_L$, tildes dropped for simplicity) are given by [bonus points: check this]

$$\begin{aligned}
\frac{\mathrm{d}s_1}{\mathrm{d}t} &= \Delta s_2 - \Gamma s_1 \\
\frac{\mathrm{d}s_2}{\mathrm{d}t} &= -\Delta s_1 - \Omega s_3 - \Gamma s_2 \\
\frac{\mathrm{d}s_3}{\mathrm{d}t} &= \Omega s_2 - \gamma(s_3 + 1)
\end{aligned} \tag{4.3}$$

where the atom+field Hamiltonian is $H_{AL} = -\hbar\Delta\sigma_3/2 + \hbar\Omega\sigma_1/2$. Find the stationary solution of these equations and discuss the excited state population as a function of detuning Δ. Using Eq.(4.2), compute the 'phase lag' of the stationary dipole moment relative to the laser field.

(3) In the lecture, we have sketched a geometric construction for the stationary state. Try to make this a little more precise in the simple case where the rotation axis corresponding to the Hamiltonian is near the 'north-south-axis' of the Bloch sphere (this requires what kind of parameters?). Focus on the $s_1 s_2$-plane near the ground state ($s_3 \approx -1$) and find the point around which the

Hamiltonian rotation happens. Try to give a geometrical construction for the stationary state in the $s_1 s_2$-plane by balancing the vector field corresponding to rotation with the vector field describing decoherence (proportional to Γ).

(4) Now upgrade these results to a macroscopic medium where the components of the Bloch vector are proportional to difference in occupation (densities) $N_e - N_g$ and the polarization field $\mathbf{P}$ (= density of electric dipole moments). Translate the 'stationary value' for the polarization field that follows from Eq.(4.2) into an electric susceptibility and the refractive index of the medium. You are allowed to consider the linear regime where the electric field (Rabi frequency) is weak and $N_e \ll N_g$. Typical numbers for the transition dipoles and density: find them on the web.

Problem 4.2 – Two-level entropy and spontaneous decay (5 points)

(1) We have seen the formula for the entropy of a two-level system whose Bloch vector has length s:

$$S = -\frac{1+s}{2}\log\frac{1+s}{2} - \frac{1-s}{2}\log\frac{1-s}{2} \qquad (4.4)$$

Make a sketch and analyze the limit of a maximally mixed state.

(2) For a two-level system that decays spontaneously, the excited state population is given by $p_e(t) = \mathrm{e}^{-\gamma t}$. We may assume that the average dipole is always zero, $\langle\sigma_1\rangle = \langle\sigma_2\rangle = 0$. Find the Bloch vector $\mathbf{s}(t)$ and plot the time-dependent entropy $S(t)$. Check that there exists a 'purification time' t_* where S is maximal and $\dot{S} < 0$ for $t > t_*$.

(3) Göran Lindblad and others have found a generic form for the equations of motion of a density operator that preserve its positivity. In the case of spontaneous decay, they tell us that

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = -\frac{\mathrm{i}}{\hbar}[H,\rho] + \mathcal{L}(\rho)\,, \qquad \mathcal{L}(\rho) = \gamma\sigma\rho\sigma^\dagger - \frac{\gamma}{2}\left\{\sigma^\dagger\sigma\rho + \rho\sigma^\dagger\sigma\right\} \qquad (4.5)$$

where $\sigma = |g\rangle\langle e| = (\sigma_1 - \mathrm{i}\sigma_2)/2$ is the 'jump down' operator and $1/\gamma$ is the excited state lifetime. By working out the traces $\mathrm{tr}\,[\sigma_a\mathcal{L}(\rho)]$ ($a = 1, 2$) show that the decoherence rate is $\Gamma = \gamma/2$ (no guarantee for the factor). This is actual a lower limit so that the characteristic times for the two-level system satisfy the rule $T_2 \le 2\,T_1$ (wikipedia: Relaxation(NMR)).

Problem 4.3 – Bloch vector images and animations (5 points)

Find on the web visualizations of Bloch vectors and their dynamics. A setting that may be particularly easy to find is a pulsed laser that rotates the Bloch over some angle. Try to make sense of the images that you see.