Einführung in die Quantenoptik I

Wintersemester 2015/16
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Übungsaufgaben Blatt 5

Ausgabe: 02. Dezember 2014
Abgabe: 16. Dezember 2014

Fr 04. Dez 15–17 Uhr – LAN-Party – Anmeldung:
LANParty@fsr.physik.uni-potsdam.de
Di 08. Dez 15–18 Uhr – KiP Cookies – Anmeldung:
kip-maphy@fsr.physik.uni-potsdam.de
Mi 09. Dez 15–18:30 Uhr – Weihnachtsfeier im KuZe

Problem 5.1 – Heisenberg picture of QM (7 points)

You remember the Hamilton operator for the 1D harmonic oscillator

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2}{2}\hat{x}^2 \tag{5.1}$$

and the classical equations of motion $m\dot{x} = p$, ...

(1) Consider the scaled coordinates $\hat{p} = \sqrt{\hbar m\omega}\,\hat{P}$, $\hat{x} = \sqrt{\hbar/(m\omega)}\,\hat{X}$ and show that $\hat{P}$, $\hat{X}$ are dimensionless and $\hat{H} = \frac{1}{2}\hbar\omega(\hat{P}^2 + \hat{X}^2)$. Work out the commutator between $\hat{P}$ and $\hat{X}$.

(2) In the Heisenberg picture of quantum mechanics, an operator $\hat{A}$ evolves in time according to the operator-valued equation of motion

$$\frac{\partial \hat{A}}{\partial t} = \frac{\mathrm{i}}{\hbar}\left[\hat{H}, \hat{A}\right] \tag{5.2}$$

Check that this reproduces the classical equations of motion for the oscillator (scaled in an elegant way).

Hint. Recall that the commutator acts on products similar to the product rule of differential calculus:

$$[H, AB] = [H, A]\,B + A\,[H, B]$$

(3) Construct the 'annihilation operator'

$$\hat{a} = \frac{\hat{X} + \mathrm{i}\hat{P}}{\sqrt{2}} \tag{5.3}$$

and remember that the commutator $[\hat{H}, \hat{a}] = -\hbar\omega\hat{a}$ translates into: 'applying the operator $\hat{a}$ lowers the energy of a state by one quantum $\hbar\omega$'. Show that its Heisenberg equation of motion is

$$\frac{\partial \hat{a}}{\partial t} = -\mathrm{i}\omega\hat{a} \tag{5.4}$$

whose solution is $\hat{a}(t) = \hat{a}(0)\,\mathrm{e}^{-\mathrm{i}\omega t}$.

Problem 5.2 – Optical field modes (7 points)

In optics and laser physics, field modes play a key role. They appear in laser beams and in Fabry-Pérot cavities, for example.

(1) Take your favorite laser beam (Gauss-Hermite or Laguerre, infrared or blue colour) and find an equation that translates the electric field $\mathbf{E}_0$ in the waist on the beam axis, the size w of the waist, the total power P and the on-axis intensity I. For an intensity of $1\,\mathrm{mW/cm}^2$, compute the number of photons per unit area and second, using the Planck-Einstein rule: 'one photon carries an energy packet hf'.

(2) Check out in the laser lab or on the web how a 'typical' Fabry-Pérot cavity looks like. Give the link between the cavity length, the round-trip time, and the free spectral range. Find out the lifetime of a photon in the cavity (keyword 'quality factor' or 'finesse'). Find a link between the intensity of a laser beam that enters the cavity and the 'circulating intensity' I_c inside. Make an estimation of the typical electromagnetic energy, for a given value of I_c. Remember the electromagnetic energy density and perform integrals by order of magnitude, using the typical dimension of the cavity mode along the cavity axis and transverse to it. Do *not* include the energy related to the mirrors. Translate into the number of photons.

Problem 5.3 – Field commutators (6 points)

In the lecture, we are going to see the Pauli-Jordan commutator between the field operators $\mathbf{E}(x)$ and $\mathbf{H}(x)$:

$$[E_j(\mathbf{x}, t),\, H_k(\mathbf{x}', t)] = (\mathrm{const.})\mathrm{i}\hbar\epsilon_{jkl}\frac{\partial}{\partial x_l}\delta(\mathbf{x} - \mathbf{x}') \tag{5.5}$$

where the constant depends on the system of units ($\pm 1/(\varepsilon_0\mu_0)$ in SI units). All other commutators vanish. All fields in this problem have to be understood as operators.

Fix the constant in Eq.(5.5) from the following 'correspondence principle': in the Heisenberg picture, the standard equation of motion for the magnetic field

$$\frac{\partial}{\partial t}\mathbf{H} = \frac{\mathrm{i}}{\hbar}\left[H,\, \mathbf{H}\right]$$

should reduce to the Faraday equation

$$\mu_0 \frac{\partial}{\partial t}\mathbf{H} = -\nabla \times \mathbf{E} \tag{5.6}$$

and the Hamiltonian H is simply the space integral over the field energy

$$H = \int \mathrm{d}V \left(\frac{\varepsilon_0}{2}\mathbf{E}^2 + \frac{\mu_0}{2}\mathbf{H}^2\right) \tag{5.7}$$

Hint. The distribution 'derivative of δ-function' in Eq.(5.5) is *defined* by the rules of partial integration. You can safely assume that there is no contribution from boundary terms.