Einführung in die Quantenoptik I

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Problem 6.1 – Jaynes-Cummings-Paul operators (5 points)

Consider a two-level atom (ladder operators $\sigma = |g\rangle\langle e|$, $\sigma^\dagger = |e\rangle\langle g|$) and a single mode of the radiation field with creation and annihilation operators $a^\dagger$, a. Their interaction can be written in the form

$$V = \hbar\Omega_1(a^\dagger\sigma + \sigma^\dagger a) \tag{6.1}$$

whose strength is set by a typical value $\Omega_1/2\pi \sim 10\,\text{MHz}$. (1) By working out the action of V on a typical state $|b, n\rangle$ ($b = e, g$, $n = 0, 1, 2\ldots$), describe in words the physical processes that it generates and comment on the name 'one-photon Rabi frequency' for Ω_1.

(2) Show that V commutes with the 'excitation operator'

$$N = a^\dagger a + \sigma^\dagger\sigma = a^\dagger a + \tfrac{1}{2}(\sigma_3 + \mathbb{1}) \tag{6.2}$$

and give a physical interpretation of the eigenstates of N.

(3) Compute the square $J^2 = (a^\dagger\sigma + \sigma^\dagger a)^2$ and try to express it via N, using the (anti)commutation relations $aa^\dagger - a^\dagger a = \mathbb{1}$, $\sigma\sigma^\dagger + \sigma^\dagger\sigma = \mathbb{1}$.

Problem 6.2 – Thermal radiation, qualitatively (8 points + 5 bonus points)

(1) Compute the average photon number at room temperature for visible light and for the frequency band used by your mobile phone.

(2) Calculate the thermal wavelength λ_T at the photosphere temperature of the Sun. In which frequency band does this fall? Check the literature or the web on the 'Wien wavelength' and the peak of the Planck spectrum. Compare spectra vs wavelength and vs frequency.

(3) Convince yourself that the energy the Earth receives from the Sun by radiation (per day, per area) is exactly balanced by the (thermal) radiation of the Earth. What is not exactly balanced is the entropy $\Delta S = \Delta Q/T$ (Clausius). Look up the key word 'solar constant' and estimate the amount of entropy the

Earth is losing into space (per day, per area). Normalized to $k_B \log 2$, this is the amount of information the Earth is receiving from the Sun. Compare over a time scale of 10^9 years with the amount of information in the genetic code of the biomass on Earth.

(4) When the Universe was about three minutes old, it was in thermal equilibrium at a temperature T_e comparable to the rest mass of the electron. Estimate this temperature and the corresponding energy density. Which particles existed in significant numbers in this era?

(5) Consider protons and neutrons at the temperature of (4) and speculate why the ratio $\exp[(m_n - m_p)c^2/k_B T_e]$ gives an idea of the ratio between proton and neutron densities. How big is the number?

(6) Somebody told you that the density of 'dark energy' in the vacuum (responsible for the accelerated expansion of the Universe, Nobel prize in physics 2011) is of the order of one proton rest mass (times c^2) per cubic meter. Find out more details on this and compare to the energy density of the cosmic microwave background.

Problem 6.3 – Vacuum energy (7 points)

In the energy of the quantized radiation field, we often 'neglect' the contribution of the zero-point energy of each field mode, given by $\frac{1}{2}\hbar\omega_k$ per mode. This 'small term' leads, in the vacuum state of the electromagnetic field, to an energy density u_{vac} that depends on the cutoff momentum k_c as follows

$$u_{\text{vac}} = C\hbar c k_c^4 \tag{6.3}$$

where C is a numerical constant 'of order unity'. (o) Calculate the constant C by counting plane wave modes in a 'quantization volume' V (two polarization modes per wave vector) and using a spherical cutoff in k-space: $|\mathbf{k}| \leq k_c$. (i) Fix the cutoff at the energy scale $E_{\text{GUT}} = \hbar c k_c$ for the 'unification' of the fundamental interactions (except gravity, look up on the Internet the keyword 'grand unified theory') and make an estimate for the corresponding vacuum energy. Express this number in proton masses (times c^2) per cubic meter. (ii) Look up the length scale $\ell_{\text{Planck}} = 1/k_c$ for the unification of gravity, relativity, and quantum theory and compare the corresponding vacuum energy to case (ii).

The vacuum energy density predicted by quantum electrodynamics (one of the simplest quantum field theories) is *very, very* different from that attributed to dark energy, and there is no good explanation yet for this.