- Introduction to General Relativity and Cosmology (Winter 2016) -

Problem set No 3

Emission 18.11.16 – Digestion 02.12.16

▷ Aufgabe 1

For any particle-like worldline $x(\tau)$ (with τ proper time) express the 4-acceleration $a^\mu(\tau) := \frac{d^2x^\mu(\tau)}{d\tau^2}$ in laboratory coordinates $(ct, \vec{x})$, and compute its square length $a_\mu a^\mu$. What components of (a^μ) do you find in the particles' momentary rest frame?

▷ Aufgabe 2 (Tensor manipulation rules)

The contraction of a single mixed tensor occurs when a pair of literal indices (one a subscript, the other a superscript) of the tensor are set equal to each other and summed over.

(a) Given a tensor of type (p, q) (that is with p upper und q lower indices). Proof that the contraction of one pair of indices yields a tensor of type $(p-1, q-1)$.

(b) Given a $n \times n$ number-scheme $(T^\alpha{}_\beta)$ for which is it known, that the contraction with an arbitrary n-vector (v^α) yields a n-vector. Proof, that by necessity the $T^\alpha{}_\beta$ are components of a $(1, 1)$-Tensor.

▷ Aufgabe 3 (Energy-Momentum Tensor)

In the GRT, the energy-momentum tensor is of paramount importance: in formulating the Einstein-Field equations it plays the role of the mass-density in the Newtonian theory of gravity.

The type and form of the energy-momentum tensor depends on the system under consideration (it can not be deduced from the GRT principles). Cold dust, for example, is characterized by a energy-momentum tensor $T^{\mu\nu} = \varrho_0 u^\mu u^\nu$, with $\varrho_0 \equiv \varrho_0(x)$ the density of mass in a local restframe at x (i.e. $\varrho_0 = \frac{\Delta m}{\Delta V}$, where Δm ist a small quantity of proper mass, and ΔV is its proper volume), and $u^\mu \equiv u^\mu(x)$ the 4-velocity of the dust at $x = (ct, \vec{x})$.

(a) Show that energy-momentum conservation $\partial_\mu T^{\mu\nu} = 0$ yields, in the non-relativistic limit, the continuity- and pressureless Euler-equations.

Cold dust is just a special case of "ideal liquids", where "ideal" means "no friction". The energy-momentum tensor of an ideal fluid reads, in a local rest frame at x

$$T^{\mu\nu}|_{\mathrm{RF}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{7}$$

where $p = p(x)$ the pressure and $\varepsilon = \varepsilon(x)$ the energy density, which besides rest-mass also includes kinetic energy, and energy of interaction.

(b) Show that in an arbitrary inertial lab-frame

$$T^{\mu\nu} = \frac{1}{c^2}(\varepsilon + p)u^\mu u^\nu + p\eta^{\mu\nu}\,, \tag{8}$$

withe $\eta = \mathrm{diag}(-1, 1, 1, 1)$ the Minkowski-tensor.

(c) Show that in the non-relativistic limit, energy-momentum conservation $\partial_\mu T^{\mu\nu} = 0$ implies the continuity- and Euler-equations of an ideal fluid.