Einführung in die Quantenoptik I

Wintersemester 2016/17
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Problem Set #7

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Problem 7.1 – Mode density, matrix elements (6 points)

In 'Fermi's Golden Rule', a formula that is often used for quantum mechanical reaction (or transition) rates, two elements appear: the 'mode density' (or 'density of states') of the end products of the reaction and the 'transition matrix element' of the operator that is responsible for the process. These two elements are discussed here for spontaneous emission.

(i) Show the following statement: in a bandwidth $\mathrm{d}\omega$ around a frequency $\omega > 0$, there are

$$\rho(\omega)\mathrm{d}\omega = \frac{\omega^2}{\pi^2 c^3}\mathrm{d}\omega \tag{7.1}$$

electromagnetic modes per unit volume. (Keyword: 'quantization box or volume'.)

(ii) For the electric field operator $\mathbf{E}(x)$ discussed in the lecture, compute the matrix element

$$\mathbf{M}_{\mathbf{k}\mu} = \langle 1_{\mathbf{k}\mu} | \mathbf{E}(x) | \mathrm{vac} \rangle \tag{7.2}$$

Argue that this quantity is proportional to the amplitude of emitting a photon with wave vector $\mathbf{k}$ and polarization label μ.

Problem 7.2 – Blackbody spectrum (7 points)

The formula of Planck for the usual spectral density of blackbody (thermal) radiation is given by

$$u(\omega) = \frac{\hbar}{\pi^2 c^3} \frac{\omega^3}{\mathrm{e}^{\beta\omega} - 1} \tag{7.3}$$

where $\beta = \hbar / k_{\mathrm{B}} T$. In this expression, one often says that the vacuum (zero-point) contribution is removed.

(i) Estimate a typical value (with units) at the Wien wavelength.

(ii) Make a plot over positive and negative frequencies, and determine the approximate behaviour in the Rayleigh-Jeans regime, $\beta|\omega| \ll 1$, and for $\omega \to \pm\infty$.

(iii) The spectrum arises in quantum optics from the Fourier transform of an autocorrelation function. As a simple model, consider the quadrature $X = a + a^\dagger$ of a single mode with frequency ω_0. Compute the Heisenberg operator $X(t)$ and the average

$$C(t - t') = \langle X(t) X(t') \rangle \tag{7.4}$$

in thermal equilibrium (average photon number $\langle a^\dagger a \rangle = \bar{n} =$ Bose-Einstein distribution. Compute the Fourier transform of $C(t - t')$ with the convention

$$S(\omega) = \int\limits_{-\infty}^{+\infty} \mathrm{d}(t - t')\, \mathrm{e}^{-\mathrm{i}\omega(t-t')} C(t - t') \tag{7.5}$$

and show that one gets two peaks with weights $\bar{n}$ and $\bar{n}+1$ at ω_0 and $-\omega_0$. What kind of autocorrelation functions would have the property that the spectrum is symmetric in ω?

Problem 7.3 – Spontaneous emission rates (7 points)

The rate of spontaneous emission for an electric dipole transition is given by (as found in the lecture)

$$\gamma_{e\to g} = \frac{|\mathbf{d}_{ge}|^2 \omega_{eg}^3}{3\pi\varepsilon_0 \hbar c^3} \tag{7.6}$$

where $\mathbf{d}_{ge}$ is the matrix element of the electric dipole moment and ω_{eg} the Bohr frequency of the transition.

(i) Make an estimate for a typical transition for an atom or molecule in the visible range. How long is the 'radiative lifetime' $1/\gamma_{e\to g}$ of the excited state?

(ii) Now switch to a magnetic dipole: the magnetic moment for an electron is of the order of the Bohr magneton $\mu_B = e\hbar/2\hbar$. Check that a handy estimate is $\mu_B/2\pi\hbar = 1.4\,\mathrm{MHz/G}$. The density of magnetic modes is the same as for electric modes up to a unit-dependent factor, so that one gets

$$\gamma_{\downarrow\to\uparrow} = \frac{\mu_0 \mu_B^2 \omega_{\downarrow\uparrow}^3}{3\pi\hbar c^3} \tag{7.7}$$

Make again an estimate for a typical Zeeman splitting frequency in a magnetic field of $10\,\mathrm{G}$, say (is this a big field?). Argue that Eq.(7.6) has to be corrected for the contribution of thermal photons.

The idea of Purcell ("Spontaneous emission probabilities at radio frequencies", *Phys. Rev.* **69** (1946) 681) of a cavity that enhances (or suppresses) spontaneous emission arose to solve the problem that nuclear transitions (keyword 'nuclear magneton') are observed, although from Eq.(7.7) the rate should be negligibly small.