- Introduction to General Relativity and Cosmology (Winter 2017) -

Problem set No 3

Emission 17.11.17 – Digestion 01.12.17

▷ Aufgabe 1

For any particle-like worldline $x(\tau)$ (with τ proper time) express the 4-acceleration $a^\mu(\tau) := \frac{d^2x^\mu(\tau)}{d\tau^2}$ in laboratory coordinates $(ct, \vec{x})$, and compute its square length $a_\mu a^\mu$. What components of (a^μ) do you find in the particles' momentary rest frame?

▷ Problem 1 (Hyperbolic motion)

Dr. Rindler moves with constant acceleration (as measured in his rest frame). He carries a good clock along (proper time τ) which allows him to determine the duration of any physical process.[1] As described with respect to an inertial system with coordinates t, x, y, z you may assume that his motion is in the x-direction.

(a) Show that in inertial coordinates Dr. Rindler's worldline may be parametrized

$$t(\tau) = \frac{c}{g}\sinh(g\tau/c) + t_0\,, \tag{5}$$

$$x(\tau) = \frac{c^2}{g}\cosh(g\tau/c) + x_0 - \frac{c^2}{g}\,, \tag{6}$$

where τ is Rindler's proper time, $\tau = 0$ corresponding to coordinate time value t_0 , and x_0 Rindler's position in the inertial system at $t = t_0$.

In special relativity, a motion with constant acceleration is sometimes called *hyperbolic motion*. Any idea why?

(b) Convince yourself that Rindler's velocity, as measured from the inertial system, is given by

$$v(t) = \frac{(t - t_0)g}{\sqrt{1 + [(t - t_0)g/c]^2}} \tag{7}$$

and thus after sufficinetly long time

$$|t - t_0| \gg c/g\,: \qquad |v(t)| \approx c\,. \tag{8}$$

For the earth gravitational acceleration $g = 9,81\text{m/sec}^2$ – what does it mean "sufficiently long" with respect to (i) the inertial system, (ii) the rest frame of Rindler?

(c) Show that in the (ct, x)-coordinates of the inertial system, the Rindler manifold of equal time events are straight lines which intersect at a spacetime event with coordinates $t_0, x_0 - c^2/g$. Without loss of generality, you may choose $t_0 = 0$ and $x_0 = c^2/g$, such that O is assigned inertial coordinates $(0, 0)$. In the xct-plane, plot the Rindler's worldline and a few of his equal time slices incl. slices where $\tau = \pm\infty$.

[1] Assume the earth to be flat, and thus its gravitational field to be homogeneous. Standing on earth, your impression is that the gravitational force pulls you downwards (the $-x$-direction). According to the equivalence principle, your impression is that of a commander of a space ship, which accelerates relative to an inertial system upwards.

(d) The event O in (c) marks a singularity in the Rindler's coordinates: for him, O is of eternal duration. Convince yourself, that from the Rindler's point of view, that singularity is always at spatial distance c^2/g. What distance would that be for the earth gravitational acceleration $g = 9,81\mathrm{m/sec}^2$?

(e) Rindler's $\tau = \pm\infty$ slices partition spacetime into four different regions. Discuss from which events Rindler can receive information, and which events can receive information from Rindler. Are there events, which are absolutely inaccessible by Rindler? If yes – what about time ordering of these events in Rindler coordinates?

(f) Rindler introduces coordinates $(c\tau, \bar{x})$ using regularly spaced spatial marks to his left and to his right, which are relative to Rindler at rest, choosing $\bar{x} = 0$ for his own position. As time standard he uses his proper time τ. Give the coordinate transformation from intertial coordinates to the Rindler coordinates.

(g) Another story: Two rockets, stacked vertically in free space, and connected by a thin rope, start with equal acceleration g. The rockets engine are such, that during boost operation, the acceleration as felt by the rocket system is constant. After a certain time T (as measured from an inertial system), there is Brennschluss for both rockets. Question: After Brennschluss – is the connecting rope still intact or not? (you may assume that the rope is infinitely thin and has no inertia). Another question: what is the difference with (f)?

▷ Aufgabe 2 (Tensor manipulation rules)

The contraction of a single mixed tensor occurs when a pair of literal indices (one a subscript, the other a superscript) of the tensor are set equal to each other and summed over.

(a) Given a tensor of type (p, q) (that is with p upper und q lower indices). Proof that the contraction of one pair of indices yields a tensor of type $(p-1, q-1)$.

(b) Given a $n \times n$ number-scheme $(T^\alpha{}_\beta)$ for which is it known, that the contraction with an arbitrary n-vector (v^α) yields a n-vector. Proof, that by necessity the $T^\alpha{}_\beta$ are components of a $(1,1)$-Tensor.

▷ Aufgabe 3 (Energy-Momentum Tensor)

In the GRT, the energy-momentum tensor is of paramount importance: in formulating the Einstein-Field equations it plays the role of the mass-density in the Newtonian theory of gravity.

The type and form of the energy-momentum tensor depends on the system under consideration (it can not be deduced from the GRT principles). Cold dust, for example, is characterized by a energy-momentum tensor $T^{\mu\nu} = \varrho_0 u^\mu u^\nu$, with $\varrho_0 \equiv \varrho_0(x)$ the density of mass in a local restframe at x (i.e. $\varrho_0 = \frac{\Delta m}{\Delta V}$, where Δm ist a small quantity of proper mass, and ΔV is its proper volume), and $u^\mu \equiv u^\mu(x)$ the 4-velocity of the dust at $x = (ct, \vec{x})$.

(a) Show that energy-momentum conservation $\partial_\mu T^{\mu\nu} = 0$ yields, in the non-relativistic limit, the continuity- and pressureless Euler-equations.

Cold dust is just a special case of "ideal liquids", where "ideal" means "no friction". The energy-momentum tensor of an ideal fluid reads, in a local rest frame at x

$$T^{\mu\nu}|_{\mathrm{RF}} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{19}$$

where $p = p(x)$ the pressure and $\varepsilon = \varepsilon(x)$ the energy density, which besides rest-mass also includes kinetic energy, and energy of interaction.

(b) Show that in an arbitrary inertial lab-frame

$$T^{\mu\nu} = \frac{1}{c^2}(\varepsilon + p)u^\mu u^\nu + p\eta^{\mu\nu}\,, \tag{20}$$

withe $\eta = \mathrm{diag}(-1, 1, 1, 1)$ the Minkowski-tensor.

(c) Show that in the non-relativistic limit, energy-momentum conservation $\partial_\mu T^{\mu\nu} = 0$ implies the continuity- and Euler-equations of an ideal fluid.