Einführung in die Quantenoptik I

Wintersemester 2017/18

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Übungsaufgaben Blatt 6 Hand out: 23 January 2018

Hand in: 01 February 2018

Aufgabe 6.1 – Transverse and longitudinal fields and sources (8 Punkte)

(1) In the lecture, we have seen an expansion of the electric and magnetic field operators into 'transverse modes'. What does this name mean? Why do we need something more when a charge density $\rho(x)$ is present?

(2) Remember the scalar potential ('C' for 'Coulomb potential')

$$\phi_C(\mathbf{x}, t) = \frac{1}{4\pi\varepsilon_0} \int \mathrm{d}V' \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|} \tag{6.1}$$

that solves the Poisson equation, $-\varepsilon_0 \nabla^2 \phi_C = \rho$. Show (from classical electrodynamics) that the shifted electric field

$$\mathbf{E}_\perp = \mathbf{E} + \nabla \phi_C \tag{6.2}$$

and the magnetic field $\mathbf{H}$ solve inhomogeneous Maxwell equations with a source term (a current density $\mathbf{j}_\perp$)

$$\begin{aligned} \varepsilon_0 \partial_t \mathbf{E}_\perp &= \nabla \times \mathbf{H} + \mathbf{j}_\perp \\ \mu_0 \partial_t \mathbf{H} &= -\nabla \times \mathbf{E}_\perp \end{aligned} \tag{6.3}$$

Give a formula for $\mathbf{j}_\perp$ and check that it is *transverse*, too, i.e., $\nabla \cdot \mathbf{j}_\perp = 0$.

(3) When these equations are used for the field operators, this scheme is often called 'quantum electrodynamics in the Coulomb gauge'. It is convenient because the scalar potential ('longitudinal field') is a degree of freedom 'enslaved' to the matter variables (the charge density $\rho(x)$). It is inconvenient because the potential $\phi_C(x)$ is non-retarded: it depends on the instantaneous value of $\rho(\mathbf{x}', t)$ even at arbitrarily large distances $\mathbf{x} - \mathbf{x}'$. Formulate an argument why for typical atomic and molecular systems, the choice of the Coulomb gauge is nevertheless a good idea.

Aufgabe 6.2 – Jaynes-Cummings-Paul model (12 Punkte + 5 bonus Punkte)

The simplest model for the interaction between atoms and light is built from one two-level atom and one field mode. This can be described by the so-called Jaynes–Cummings–Paul Hamiltonian

$$H_{\mathrm{JCP}} = \hbar\omega_A \sigma^\dagger \sigma + \hbar\omega_c a^\dagger a - \mathbf{d}\cdot\mathbf{E}(\mathbf{r}_A) \tag{6.4}$$

where σ and $\sigma^\dagger$ are the 'jump down' and 'jump up' operators for the atom. We call them dipole operators in the following. The 'cavity frequency' ω_c gives the photon energy of the field mode, and $\mathbf{E}(\mathbf{r})$ contains just one pair a, $a^\dagger$ of photon operators.

(1) Write down the interaction Hamiltonian V using the atomic dipole operators σ, $\sigma^\dagger$ and the photon operators a, $a^\dagger$. Evaluate its matrix elements $\langle e, 1|V|g, 0\rangle$ and $\langle g, 1|V|e, 0\rangle =: \hbar g$. (No confusion between the atomic ground state $|g\rangle$ and the coupling constant g.)

(2) [5 Bonus points] In quantum optics, the resonance (or 'rotating-wave') approximation is often appropriate. Remember that in this approximation, one works with the simplified interaction

$$-\mathbf{d}\cdot\mathbf{E}(\mathbf{r}_A) \approx V_{\mathrm{JCP}} = \hbar\left(g a^\dagger \sigma + g^* \sigma^\dagger a\right) \tag{6.5}$$

Write down the two terms that are neglected here and speculate why this is a good approximation. (**Hint**: consider the free time evolution of the dipole and photon operators. You can also consider the 'elementary processes' related to the absorption and emission of a photon.)

(3) Show that the only non-zero matrix elements of the interaction (6.5) are:

$$\langle g, n+1|V_{\mathrm{JCP}}|e, n\rangle = \hbar g\sqrt{n+1}, \qquad \langle e, n|V_{\mathrm{JCP}}|g, n'\rangle = \ldots \tag{6.6}$$

and conclude that the JCP Hamiltonian can be diagonalized exactly in the subspaces spanned by $|e, n\rangle$ and $|g, n+1\rangle$ (for $n = 0, 1, 2\ldots$). The ground state $|g, 0\rangle$ remains 'alone'.

(4) Consider the simple case of an initial state $|e, n\rangle$ with exactly n photons and the atom in the excited state, and assume perfect resonance $\omega_A = \omega_c$. Show that Rabi oscillations occur with a frequency $g\sqrt{n+1}$. ($n = 0$: 'vacuum Rabi oscillations'.)

(5) In so-called coherent states, the photon number is distributed around a mean value $\bar{n}$ with a typical width $\Delta n = \bar{n}^{1/2}$. Argue that the amplitude of the Rabi oscillations decays because the different photon numbers get out of phase and justify the estimation $|g|\tau \sim 1$ for the typical decay time τ.

This decay is called 'collapse' of Rabi oscillations. The envelope of the oscillations does not follow an exponential, but approximately a gaussian if $\bar{n}$ is large enough. At even larger times $|g|t \sim \bar{n}^{1/2}$, the Rabi oscillations 'revive' ('collapse and revival').