Chapter 2

Asymptotic analysis

2.1 Introduction

In this chapter, we review some elements of asymptotic analysis of functions and differential equations. In fact, this material has been used implicitly in the previous sections, when limits of special functions were discussed. The WKB approximation itself may be viewed as an asymptotic expansion of the SCHRÖDINGER equation solutions around the singular point '$\hbar = 0$'.

2.1.1 Quantify limiting behaviours

To say that a function $f(x)$ vanishes for $x \to 0$, say, is often not enough information. We also want to know 'how' this limit is approached. A useful class of functions that all vanish with x are the powers. We say that 'f is of order x^n' ($n \in \mathbb{N}$) if the series expansion of $f(x)$ around the origin starts with this power:

$$x \to 0: \quad f(x) = \mathcal{O}(x^n) \Leftrightarrow \lim_{x \to 0} \frac{f(x)}{x^n} = A, \ |A| < \infty$$

Note that the coefficient A is uniquely determined by this prescription.

The powers are not sufficient to classify all asymptotic behaviours. There are functions that vanish 'faster than any power of x', for example

$f(x) = \mathrm{e}^{-1/x}$. Using DE L'HÔPITAL's rule, one finds indeed

$$\lim_{x\to 0} \frac{\mathrm{e}^{-1/x}}{x^n} = \lim_{z\to\infty} \frac{z^n}{\mathrm{e}^z} = \lim_{z\to\infty} \frac{n!}{\mathrm{e}^z} = 0$$

We can thus add another class of functions $\exp(-a/x^\nu)$ with $a > 0$, $\nu > 0$ to the powers.

What happens if $f(0) = \infty$? To describe this divergence, we can use the inverse powers x^{-n} as well as the exponential $\mathrm{e}^{1/x}$ (diverges faster than any power). Now there are also functions like the logarithm $\log(1/x)$ that diverge more slowly than any inverse power of x:

$$\lim_{x\to 0} \frac{\log(1/x)}{x^{-\sigma}} = \lim_{z\to\infty} \frac{\log z}{z^\sigma} = \lim_{z\to\infty} \frac{1}{\sigma z^\sigma} = 0, \ (\sigma > 0)$$

We thus have a sort of hierarchy of diverging functions

$$x \to 0 : \ldots \mathrm{e}^{1/x^2} > \mathrm{e}^{1/x} > \ldots > \frac{1}{x^2} > \frac{1}{x} > \ldots > \log(1/x) > \ldots$$

and a similar hierarchy for functions that vanish at $x = 0$.

2.1.2 Asymptotic series

These are series that typically diverge (their convergence radius is zero) but give nevertheless highly accurate results when a finite number of terms is retained. To see how is this possible, consider the following integral (the 'STIELTJES integral'):

$$I(\omega) = \int_0^\infty \frac{\omega \mathrm{e}^{-x}\,\mathrm{d}x}{\omega + x} \tag{2.1}$$

with ω a large parameter. If we put simply $\omega = \infty$, we get

$$I(\infty) = \int_0^\infty \mathrm{e}^{-x}\,\mathrm{d}x = 1$$

This suggests that we expand the factor $\omega/(\omega+x) = (1+x/\omega)^{-1}$ in a power series in x/ω:

$$\frac{1}{1+x/\omega} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{\omega^n}$$

In fact, the exponential e^{-x} in the integrand implies that only values in the range $0 \le x \lesssim 1$ contribute to the integral (see also fig.2.1). The ratio x/ω

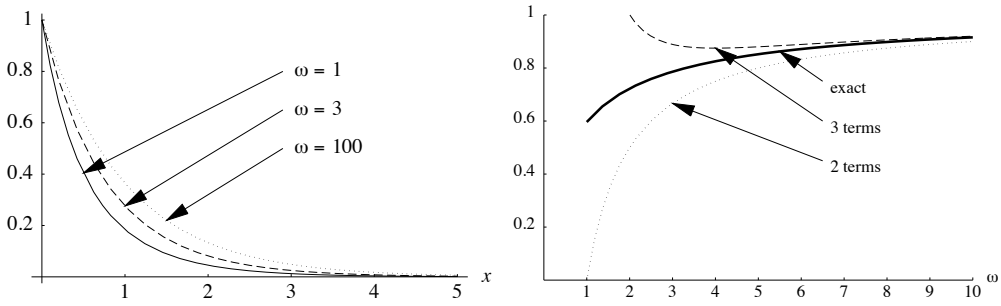

Figure 2.1: Left graph: integrand of the integral (2.1) for several values of the parameter ω.
Right graph: value of the integral $I(\omega)$.

is then of order $\mathcal{O}(1/\omega)$ and small in the limit $\omega \to \infty$. The series expansion made above should therefore give a good approximation. Integrating term by term, we find the series

$$I(\omega) = \sum_{n=0}^{\infty} \frac{(-1)^n n!}{\omega^n}.$$

This is a strange result: the convergence radius of this series is zero, as we may check from the ratio test

$$\lim_{n\to\infty} \frac{(-1)^n n!\, \omega^{n-1}}{(-1)^{n-1}(n-1)!\, \omega^n} = \lim_{n\to\infty} \frac{n}{\omega} = \infty$$

(for fixed ω). However, we can estimate the remainder if we truncate the series at a fixed number N of terms. Using the summation formulas for a geometric series, we have the identity

$$\frac{\omega}{\omega + x} = \sum_{n=0}^{N} \frac{(-1)^n x^n}{\omega^n} + \frac{(-x)^{N+1}}{\omega^N(\omega + x)}$$

and the contribution of the last term to the integral may be overestimated using $(\omega + x)^{-1} < x^{-1}$:

$$|R_N(\omega)| = \left| \int_0^\infty \frac{(-x)^{N+1} \mathrm{e}^{-x}}{\omega^N(\omega + x)} \right| < \int_0^\infty \frac{x^N \mathrm{e}^{-x}}{\omega^N} = \frac{N!}{\omega^N}$$

(One may even show (using the definition given above) that the error $R_N(\omega)$ is of order $\mathcal{O}(\omega^{-N-1})$.)

We thus find that although the series does not converge, the truncation error for finite N can be made arbitrarily small provided the parameter ω is large. For a fixed large value of ω, one therefore obtains accurate results with a finite number of terms. Typically, one computes only the first few terms of the asymptotic series. The first neglected term then allows to estimate the error made. Figure 2.1 shows the accuracy obtained when the first two or three terms of the asymptotic series are taken.

We can now state the general definition of an **asymptotic series**. The function $f(x)$ has an asymptotic series expansion around x_0,

$$f(x) \sim \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

if and only if the truncation error is of the same order as the first neglected term:

$$f(x) = \sum_{n=0}^{N-1} a_n (x - x_0)^n + \mathcal{O}\left((x - x_0)^N\right)$$

Note that by definition, a convergent power series is also an asymptotic series. The reverse is not true, as we saw in the preceding example. A more general type of asymptotic expansions involves a generic series whose terms are functions $a_n g_n(x)$ where the g_n are not necessarily powers, but, e.g., a product $x^n \, \mathrm{e}^{-1/x}$.

Example: two series for the BESSEL functions

We shall show below that the Bessel function $J_0(x)$ has the following series expansion around the origin:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x^2}{4}\right)^n = 1 - \frac{x^2}{4} + \mathcal{O}(x^4) \tag{2.2}$$

On the other hand, for large values of x, the following asymptotic expansion holds

$$\begin{aligned} x \to \infty : & \\ J_0(x) &\sim \sqrt{\frac{2}{\pi x}} \left(u(x) \cos(x - \tfrac{\pi}{4}) + v(x) \sin(x - \tfrac{\pi}{4})\right) \qquad (2.3) \\ u(x) &= \sum_{n=0}^{\infty} \frac{(-1)^n ((4n-1)!!)^2}{(2n)! \, (8x)^{2n}} = 1 - \frac{9}{128 x^2} + \mathcal{O}(1/x^4) \end{aligned}$$

$$v(x) \quad = \quad \sum_{n=0}^{\infty} \frac{(-1)^n((4n+1)!!)^2}{(2n+1)!\,(8x)^{2n+1}} = \frac{1}{8x} + \mathcal{O}(1/x^3)$$

Using the ratio test, one sees easily that the power series (2.2) converges for all values of x (its convergence radius is infinite). On the other hand, the asymptotic expansion (2.3) has zero convergence radius.

In fig.2.2, we compare the two expansions to the exact function $J_0(x)$. The first two terms of the power series (thin line) describes $J_0(x)$ only for $x \lesssim 1$. The dotted line gives the first term of the asymptotic series and is already quite accurate in the range $x \gtrsim 1$.

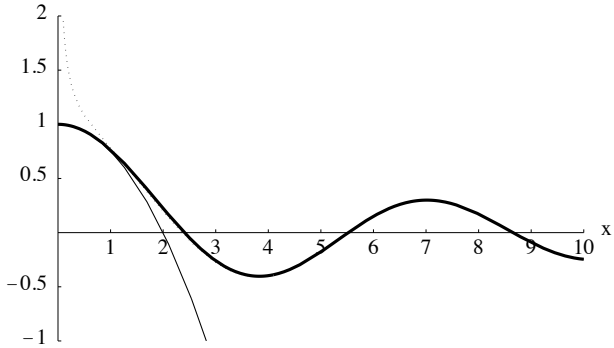

Figure 2.2: Asymptotic forms of the BESSEL function $J_0(x)$. Thick solid line: $J_0(x)$, thin solid line: first two terms of the power series at $x = 0$, dotted line: first term of asymptotic series for large x.

2.2 Asymptotic solutions of differential equations

The case mentioned above where we performed an asymptotic expansion of an integral is typically the easier one. Often one only knows the differential equation satisfied by the function $f(x)$. We discuss in this subsection how asymptotic series may be obtained in that case. We focus on linear, ordinary differential equations with variable coefficients. (See Nayfeh (1981, chaps.1, 13) for a more detailed treatment with more examples.)

2.2.1 First-order equations

The general form of a first order equation is

$$y' = F(x)y \tag{2.4}$$

where $y' \equiv \mathrm{d}y/\mathrm{d}x$ and $F(x)$ is a given function. The basic idea to get an asymptotic series for $y(x)$ is to expand the coefficient $F(x)$ in a power series and to compare the coefficients of like powers.

A point x_0 where $F(x)$ allows for a convergent series expansion is called an *ordinary point* of the differential equation. The solution $y(x)$ can then be obtained in terms of a convergent power series around x_0. If $F(x)$ diverges at $x = x_0$, we have a *singular point*. The solution then has a different structure, need not be finite, and its series expansion need not be convergent.

These cases are most easily illustrated using the exact solution to (2.4):

$$y(x) = \exp\left(\int^x F(x')\,\mathrm{d}x'\right) \tag{2.5}$$

Around an ordinary point x_0, we may expand $F(x)$ and find, after integrating the power series term by term

$$y(x) = \exp\left(C + \sum_{n=0}^{\infty} \frac{a_n}{n+1}(x - x_0)^{n+1}\right)$$

The exponential function has infinite convergence radius, and therefore we get a convergent series expansion for $y(x)$ at x_0.

Things change when $F(x)$ has a singularity at x_0. Suppose that we have a simple pole

$$F(x) = \frac{F_0}{x - x_0} + \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

The solution (2.5) then contains a factor of the form

regular singular point:

$$y(x) = (x - x_0)^{F_0} \sum_{n=0}^{\infty} b_n (x - x_0)^n$$

where the exponent of the first factor is possibly non-integer and/or negative. This solution therefore may also diverge at x_0, but at most like a power law. Such a point x_0 is called a *regular singular point*.

Finally, if $F(x)$ has a pole of second order, we find a solution

irregular singular point:

$$y(x) = \mathrm{e}^{-F_0/(x-x_0)}(x-x_0)^{F_1}\sum_{n=0}^{\infty} b_n(x-x_0)^n$$

that contains an exponential factor with an essential singularity at x_0. Depending on the sign of F_0, the solution vanishes or diverges when x approaches x_0 from above. This type of behaviour is characteristic for $F(x)$ having a pole of second order or higher at x_0. Such a point is called an *irregular singular point*.

The kind of solutions obtained for first-order differential equations will also appear for equations of second order. We shall discuss examples in the following subsection.

2.2.2 Second-order differential equations

Their general form is similar to the SCHRÖDINGER equation

$$y'' + p(x)y' + q(x)y = 0 \tag{2.6}$$

The classification of a point x_0 is as follows:

- *ordinary point*: $p(x_0)$ and $q(x_0)$ are finite and have convergent power series. At least one solution can then be obtained as a power series.

- *regular singular point*: $p(x)$ has at most a pole of first order and $q(x)$ has at most a pole of second order. At least one solution is then of the form of a fractional power times a series

  $$y_1(x) = x^{\sigma}\sum_{n=0}^{\infty} b_n(x-x_0)^n$$

  where σ is called the 'index'. A series of this form is called a 'FROBENIUS *ansatz*'.

- *irregular singular point*: $p(x)$ or $q(x)$ have higher-order singularities than those mentioned above. At least one solution is then of the form of

  $$y_1(x) = \mathrm{e}^{-\lambda/(x-x_0)^{\nu}}x^{\sigma}\sum_{n=0}^{\infty} b_n(x-x_0)^n$$

with possibly an essential singularity at x_0.

To illustrate these behaviours, we shall derive the series (2.2, 2.3) for the BESSEL function $J_0(x)$ given above.

2.2.3 Example: Bessel function

Definition of $J_0(x)$. The BESSEL function is the physical solution to the SCHRÖDINGER equation for a free particle in two dimensions with zero angular momentum around the origin. The radial equation then reads

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}r^2} - \frac{\hbar^2}{2m}\frac{1}{r}\frac{\mathrm{d}\psi}{\mathrm{d}r} = \frac{\hbar^2 k^2}{2m}\psi \tag{2.7}$$

where k is the wave number of the particle. Using the scaled coordinate $x = kr$, we find the BESSEL equation ($y(x) = \psi(r)$):

$$y'' + \frac{1}{x}y' + y = 0 \tag{2.8}$$

Since $p(x) = 1/x$, the origin $x_0 = 0$ is a regular singular point.

Power series for small x. Using the FROBENIUS *ansatz* for a power series

$$y_1(x) = \sum_{n=0}^{\infty} a_n x^{\sigma+n},$$

we find the following conditions

$$\sigma^2 a_0 = 0 \tag{2.9a}$$

$$(\sigma+1)^2 a_1 = 0 \tag{2.9b}$$

$$a_{n+2} = -\frac{a_n}{(\sigma+n+2)^2} \tag{2.9c}$$

We choose the solution $a_0 \neq 0$, and get $\sigma = 0$ and $a_1 = 0$. (This solution is finite at the origin. If we had chosen $a_1 \neq 0$, then $\sigma = -1$ and $a_0 = 0$. But this would give the same power series, starting with $x^{-1}a_1 x = a_1$.) We obtain the power series

$$y_1(x) = a_0 x^\sigma \left(1 - \frac{x^2}{(\sigma+2)^2} + \mathcal{O}(x^4)\right) \tag{2.10}$$

One may verify that this function (for arbitrary σ) satisfies

$$y'' + \frac{1}{x}y' + y = a_0\sigma^2 x^{\sigma-2}$$

To get a second, linear independent solution, we differentiate this equation with respect to σ:

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial y}{\partial \sigma} + \frac{1}{x}\frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial y}{\partial \sigma} + \frac{\partial y}{\partial \sigma} = 2a_0\sigma x^{\sigma-2} + a_0\sigma^2 x^{\sigma-2}\log x$$

For $\sigma = 0$, the derivative $\partial y/\partial\sigma$ therefore also satisfies the BESSEL equation.

Differentiating the series (2.10) with respect to σ, we find a second solution $y_2(x)$:

$$y_2(x) = y_1(x)\log x + \frac{x^2}{4} + \mathcal{O}(x^4)$$

that has a logarithmic singularity at the origin.

We get the BESSEL function $J_0(x)$ by putting the coefficient $a_0 = 1$. One then gets the series (2.2). The other BESSEL function $Y_0(z)$ is a linear combination of $y_2(x)$ and $J_0(x)$.

Asymptotic series for large x. We now want to obtain a series for $J_0(x)$ in the vicinity of $x = \infty$. What kind of point is this in the differential equation? To give an answer, we use the variable $z = 1/x$ and obtain:

$$\begin{aligned}
\frac{\mathrm{d}y}{\mathrm{d}x} &= -\frac{1}{x^2}\frac{\mathrm{d}y}{\mathrm{d}z} = -z^2\frac{\mathrm{d}y}{\mathrm{d}z} \\
\frac{\mathrm{d}^2y}{\mathrm{d}^2x} &= \frac{1}{x^4}\frac{\mathrm{d}^2y}{\mathrm{d}^2z} + \frac{2}{x^3}\frac{\mathrm{d}y}{\mathrm{d}z} = z^4\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + 2z^3\frac{\mathrm{d}y}{\mathrm{d}z}
\end{aligned}$$

The differential equation is thus transformed into

$$\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + \left(\frac{2}{z} - \frac{p(1/z)}{z^2}\right)\frac{\mathrm{d}y}{\mathrm{d}z} + \frac{q(1/z)}{z^4}y = 0$$

The point $x = \infty$ (or $z = 0$) is regular when the functions

$$2x - x^2p(x), \quad x^4q(x)$$

are finite at infinity. If the first diverges at most like x and the second at most like x^2, then infinity is a regular singular point. Otherwise, one has an irregular singular point at infinity.

For the BESSEL equation, there is an irregular singular point at infinity because of $q(x) = 1$. We are therefore forced to make the following *ansatz* for the solutions

$$\mathrm{e}^{\lambda x^\nu} \sum_{n=0}^{\infty} a_n x^{-\sigma-n}$$

Note that we still have to determine the unknown coefficients in the first exponential factor.

Putting this power series into the differential equation, we get

$$\begin{aligned}
&\left[\lambda^2\nu^2 x^{2\nu-2} + \lambda\nu^2 x^{\nu-2} + 1\right] \sum_{n=0}^{\infty} a_n x^{-n} + \\
&+ \left[2\lambda\nu x^{\nu-2} + x^{-2}\right] \sum_{n=0}^{\infty} a_n(-\sigma-n)x^{-n} + \\
&+ x^{-2} \sum_{n=0}^{\infty} a_n(-\sigma-n)(-\sigma-n-1)x^{-n} = 0 \qquad (2.11)
\end{aligned}$$

We now identify the 'most diverging term' in this expression. The case $\nu \leq 0$ can be excluded because then the exponential $\mathrm{e}^{\lambda x^\nu}$ could have been expanded in a convergent power series. For $\nu > 0$, the most diverging power is $2\nu - 2$ that can only be balanced by the term 1 in the first line. We thus find the exponent $\nu = 1$ and furthermore

$$\lambda^2\nu^2 + 1 = 0, \quad \text{hence } \lambda = \pm\mathrm{i}.$$

The next-to-leading order in (2.11) is the power $x^{\nu-2} = x^{-1}$. Its coefficient vanishes if

$$(1 - 2\sigma)\, a_0 = 0$$

and we get $\sigma = \frac{1}{2}$ if we look for a solution with $a_0 \neq 0$. Finally, for the powers x^{-2-n}, we get the following recurrence relation

$$a_{n+1} = a_n \frac{(\sigma+n)^2}{\lambda(2\sigma+2n+1)}$$

that may be solved using $\lambda = \mathrm{i}, \sigma = \frac{1}{2}$, $a_0 = 1$ to give

$$a_n = \frac{(-\mathrm{i})^n\, ((2n-1)!!)^2}{8^n\, n!}, \quad n \geq 1.$$

If we had taken $\lambda = -\mathrm{i}$, we would have obtained the complex conjugate.

We thus get a complex series (whose convergence radius is zero) and may take its real and imaginary part to describe two independent asymptotic solutions to the BESSEL equation:

$$y(x) = y_1(x) + \mathrm{i}y_2(x) \sim \frac{\mathrm{e}^{\mathrm{i}x}}{\sqrt{x}} \sum_{n=0}^{\infty} \frac{a_n}{x^n} \tag{2.12}$$

The power series *ansatz* cannot determine, however, which linear combination of $y_{1,2}(x)$ we have to take to describe asymptotically the BESSEL function $J_0(x)$. To this end, we have to find an alternative asymptotic form of $J_0(x)$ for large x, starting, e.g., from an integral representation.

We introduced above $J_0(kr)$ as the physically acceptable solution to the radial SCHRÖDINGER equation (2.7) for a free particle in two dimensions with angular momentum zero. Note that we can also get this solution by expanding a plane wave into angular momentum eigenfunctions:

$$\mathrm{e}^{\mathrm{i}ky} = \sum_{m=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}m\varphi} J_m(kr)$$

where $y = r\sin\varphi$ in polar coordinates. Since this relation is simply a FOURIER series, its inversion is given by a FOURIER integral over a period of the angle φ:

$$J_0(kr) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \mathrm{d}\varphi\, \mathrm{e}^{\mathrm{i}kr\sin\varphi} \tag{2.13}$$

We can check that the normalisation is correct by putting $kr = 0$ (and getting $J_0(0) = 1$).

The integral representation (2.13) is taylormade to get the asymptotic behaviour for large $kr = x$ using the method of stationary phase. The stationary points of the phase $x\sin\varphi$ are located at $\varphi_{1,2} = \pm\pi/2$. Expanding the phase to second order around these points, we find

$$\begin{aligned} x \to \infty: \quad J_0(x) &= \frac{1}{2\pi}\left(\mathrm{e}^{\mathrm{i}x} \int_{-\infty}^{+\infty} \mathrm{d}\varphi\, \mathrm{e}^{-\mathrm{i}x(\varphi-\varphi_1)^2/2} + \text{c.c.}\right) \\ &= \frac{1}{\sqrt{2\pi x}}\left(\mathrm{e}^{\mathrm{i}x}\mathrm{e}^{-\mathrm{i}\pi/4} + \text{c.c.}\right) \end{aligned} \tag{2.14}$$

The coefficient in front of the asymptotic solution $y(x)$ in (2.12) is therefore equal to $1/\sqrt{2\pi\mathrm{i}}$. Including the higher-order terms of the power series (that could also be obtained from a systematic investigation of the corrections to

the stationary phase integral), we get the following asymptotic representation for the BESSEL function:

$$J_0(x) = \frac{1}{\sqrt{2\pi}} \left(\mathrm{e}^{-\mathrm{i}\pi/4} y(x) + \mathrm{e}^{\mathrm{i}\pi/4} y^*(x) \right)$$

The explicit asymptotic series is thus

$$\begin{aligned} J_0(x) \quad \sim \quad & \sqrt{\frac{2}{\pi x}} \Bigg(\cos\left(x - \tfrac{\pi}{4}\right) \sum_{n=0}^{\infty} \frac{a_{2n}}{x^{2n}} + \\ & + \sin\left(x - \tfrac{\pi}{4}\right) \sum_{n=0}^{\infty} \frac{-\mathrm{Im}\, a_{2n+1}}{x^{2n+1}} \Bigg) \end{aligned} \tag{2.15}$$

One checks immediately that this series is identical to (2.3).

Bibliography

A. H. Nayfeh (1981). *Introduction to Perturbation Techniques*. Wiley, New York.