Asymptotische Methoden in der Wellenmechanik

Carsten Henkel — Wintersemester 2018/19

Übungsblatt 4 — Abgabe: 12. Dezember 2018

Aufgabe 4.1 – An electron fluid bouncing off a wall (12 Punkte)

In this problem, you are invited to study a problem of a metallic surface where optics and hydrodynamics come together.

(1) We describe the light field by a scalar potential, $\mathbf{E} = -\nabla\phi$ and set up a reflection/transmission problem by writing:

$$z < 0: \qquad \phi(\mathrm{r}, t) = \mathrm{e}^{\mathrm{i}(kx-\omega t)} \left(\mathrm{e}^{-\kappa z} + R\,\mathrm{e}^{\kappa z}\right) \tag{4.1}$$

Find reasons why R may be called a “reflection coefficient”. Assuming that the half-space $z < 0$ is filled with vacuum, show that $\kappa = k$.

(2) We consider a metal that fills the half-space $z > 0$. For the electronic current density $\mathbf{j}$, one can use the “hydrodynamic approximation” which corresponds to the equation of motion

$$\frac{\partial}{\partial t}\mathbf{j} + \frac{1}{\tau}\mathbf{j} = \frac{e\rho_0}{m}\mathbf{E} - \beta^2\nabla\rho \tag{4.2}$$

This is linearised equation for small oscillations ρ of the charge density around its equilibrium value ρ_0. The time τ is the typical scattering time, the velocity β is of the order of the Fermi velocity. We define the plasma frequency $\Omega^2 = e\rho_0/(m\varepsilon_0)$. For a metal like aluminum or gold, find typical parameters and compute the length scales

$$\text{mean free path:} \qquad \ell = \beta\tau \tag{4.3}$$

$$\text{screening length:} \qquad \Lambda = \beta/\Omega \tag{4.4}$$

(3) On length scales large compared to these two, one can neglect the last term in Eq.(4.2) and get Ohm’s law $\mathbf{j}(\mathbf{r}, \omega) = \sigma(\omega)\mathbf{E}(\mathbf{r}, \omega)$ which provides a local relation between the current density and the electric field. Compute $\sigma(\omega)$ and show that $\varepsilon_0\Omega^2\tau$ is the DC conductivity. Are the parameters that you found before consistent with the measured conductivity?

(4) We now take into account the screening length and solve Eq.(4.2) for a field (current, charge density, ...) that varies like $\mathrm{e}^{\mathrm{i}(kx-\omega t)}$. Find two coupled equations

for the z-dependent Fourier components of the charge density $\rho(z)$ and the electric potential:

$$z > 0: \qquad \phi(\mathbf{r}, t) = \phi(z)\, \mathrm{e}^{\mathrm{i}(kx - \omega t)} \tag{4.5}$$

(5) One needs an "additional boundary condition" (ABC) to complete the continuity relations for the field and the charges:

- the potential $\phi(z)$ is continuous across the surface $z = 0$
- the normal current density $j_z(z) \to 0$, as one approaches the surface from inside $z \to 0^+$.

A suitable *Ansatz* to solve the coupled equations is based on a linear combination of exponentials $A\,\mathrm{e}^{-kz} + B\,\mathrm{e}^{-qz}$ for both fields, where

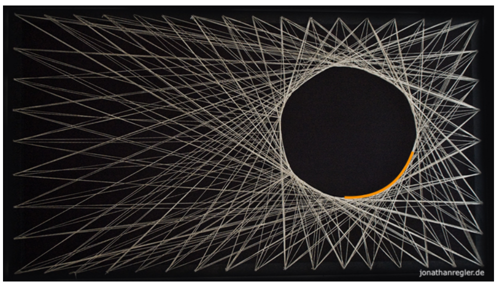
$$\omega^2 + \mathrm{i}\frac{\omega}{\tau} = \Omega^2 + \beta^2(k^2 + q^2) \tag{4.6}$$

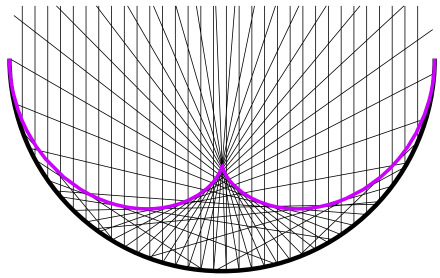
Justify this equation and find the ratio A/B by applying the ABC. What happens in the limit $\beta \to 0$? Compute the average oscillating charge Q and its "centroid" d:

$$Q = \int\limits_0^\infty \mathrm{d}z\, \rho(z)\,, \qquad d = \frac{1}{Q}\int\limits_0^\infty \mathrm{d}z\, z\rho(z) \tag{4.7}$$

Aufgabe 4.2 – Caustics: "light and fire" (8 Punkte)

These pictures illustrate the idea of a caustic: "natural focussing of light" or a place where many light rays meet. (What is the origin of the word?)





Along the orange curve in the left picture, a family of rays going from the lower to the right edge meet. Assume that the ray end points are displaced in a linear fashion, can you compute the limiting curve? Is it really part of a circle? For the "coffee cup caustic" on the right side, set up a computer program to draw the rays. Try to compute the position of the "cusp" where the two curved caustics meet. How do the two curves behave in its vicinity?

Solution. Left picture: the family of rays can be written as

$$y(x; s) = y_0 + s\alpha - \frac{y_0 + s\alpha}{x_0 + s}(x - x_0 - s) \tag{4.8}$$

where x_0 and y_0 are reference positions on the x- and y-axes, α gives the ratio between steps (or nails) on the axes, and the parameter s gives the crossing with the x-axis.

We find the caustic by taking two rays with neighbouring values s and $s{+}ds$ and computing their crossing point. By taking the difference of the two equations, one gets, in the limit $ds \to 0$ the derivative of Eq.(4.8) with respect to the parameter s. By setting this to zero, we get a second equation that we solve together with the ray (4.8).

The resulting equation for the caustic can be brought into the form

$$y(x) = 2(y_0 - \alpha x_0) - x\alpha - \sqrt{8x\alpha\,(\alpha x_0 - y_0)} \tag{4.9}$$

which is a combination of a linear function and a square root, probably not a circle. This is plotted in blue ($x_0 = -10$, $y_0 = 6$, $\alpha = \frac{3}{5}$).